# ABSTRACT

Title of dissertation: STRONG SHIFT EQUIVALENCE, ALGEBRAIC K-THEORY, AND ISOLATING ZERO-DIMENSIONAL DYNAMICS ON MANIFOLDS

Scott Schmieding, Doctor of Philosophy, 2016

Dissertation directed by: Professor Mike Boyle
Department of Mathematics

We study the relations of shift equivalence and strong shift equivalence for matrices over a ring $\mathcal{R}$, and establish a connection between these relations and algebraic K-theory. We utilize this connection to obtain results in two areas where the shift and strong shift equivalence relations play an important role: the study of finite group extensions of shifts of finite type, and the Generalized Spectral Conjectures of Boyle and Handelman for nonnegative matrices over subrings of the real numbers.

We show the refinement of the shift equivalence class of a matrix $A$ over a ring $\mathcal{R}$ by strong shift equivalence classes over the ring is classified by a quotient $NK_1(\mathcal{R})/E(A,\mathcal{R})$ of the algebraic K-group $NK_1(\mathcal{R})$. We use the K-theory of non-commutative localizations to show that in certain cases the subgroup $E(A,\mathcal{R})$ must vanish, including the case $A$ is invertible over $\mathcal{R}$.

We use the K-theory connection to clarify the structure of algebraic invariants for finite group extensions of shifts of finite type. In particular, we give a strong negative answer to a question of Parry, who asked whether the dynamical zeta

function determines up to finitely many topological conjugacy classes the extensions by $G$ of a fixed mixing shift of finite type.

We apply the K-theory connection to prove the equivalence of a strong and weak form of the Generalized Spectral Conjecture of Boyle and Handelman for primitive matrices over subrings of $\mathbb{R}$.

We construct explicit matrices whose class in the algebraic K-group $NK_1(\mathcal{R})$ is non-zero for certain rings $\mathcal{R}$ motivated by applications.

We study the possible dynamics of the restriction of a homeomorphism of a compact manifold to an isolated zero-dimensional set. We prove that for $n \geq 3$ every compact zero-dimensional system can arise as an isolated invariant set for a homeomorphism of a compact $n$-manifold. In dimension two, we provide obstructions and examples.

# STRONG SHIFT EQUIVALENCE, ALGEBRAIC K-THEORY, AND ISOLATING ZERO-DIMENSIONAL DYNAMICS ON MANIFOLDS

by

Scott Schmieding

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Advisory Committee:
Professor Mike Boyle, Chair/Advisor
Professor Giovanni Forni
Professor Michael Jakobson
Professor Jonathan Rosenberg
Professor James Purtilo

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Table of Contents

# List of Figures

# Chapter 1: Introduction

In the introduction, we discuss the separate chapters of this thesis. Chapters 2, 3, 4, and 6 contain joint work with Mike Boyle. Chapter 3 is to appear (with tiny modifications) in Ergodic Theory and Dynamical Systems. Chapter 4 is to appear (with tiny modifications) in Linear Algebra and its Applications. Chapter 2 has been conditionally accepted by another journal. Finally, we note that the Chapters 3 and 4 appealed to stronger statements in an earlier draft of Chapter 2, and the corresponding corrections are discussed at the end of Section 2 in Chapter 2.

## 1.1 Strong shift equivalence and algebraic $K$-theory

Let $\mathcal{R}$ (always assumed to contained 0 and 1) be a subset of a ring. Let $A, B$ be square matrices over $\mathcal{R}$. The matrices $A, B$ are said to be *elementary strong shift equivalent* over $\mathcal{R}$ (ESSE-$\mathcal{R}$) if there exist matrices $U, V$ over $\mathcal{R}$ such that $A = UV$ and $B = VU$. Note that $A$ and $B$ are not required to be of the same size. We say $A$ and $B$ are *strong shift equivalent* over $\mathcal{R}$ (SSE-$\mathcal{R}$) if there exists a chain of elementary strong shift equivalences connecting $A$ and $B$. The relation SSE-$\mathcal{R}$ is an equivalence relation, and is the transitive closure of ESSE-$\mathcal{R}$. We say $A$ and $B$ are *shift equivalent* over $\mathcal{R}$ (SE-$\mathcal{R}$) if there exists matrices $U, V$ over $\mathcal{R}$ and $l \in \mathbb{N}$ such that the following hold:

$$A^l = UV, B^l = VU$$

$$AU = UB, VA = BV$$

If $A, B$ are SSE-$\mathcal{R}$, then they are SE-$\mathcal{R}$.

The shift equivalence and strong shift equivalence relations were originally introduced by Williams in the context of dynamical systems. Their importance to symbolic dynamics in particular became apparent with Williams' foundational result [1] classifying subshifts of finite type up to topological conjugacy by strong shift equivalence of matrices presenting them over the semi-ring $\mathbb{Z}_+$. We consider the following question:

Question 1: For any $\mathcal{R}$, when does SE-$\mathcal{R}$ determine SSE-$\mathcal{R}$?

This question is one of the primary investigations of this thesis. The answer to Question 1 was shown to be yes in the following cases:

1. $\mathcal{R} = \mathbb{Z}$ due to Williams in the 70s

2. $\mathcal{R}$ is a principal ideal domain, due to Effros in 1981 [2]

3. $\mathcal{R}$ is a Dedekind domain, due to Boyle & Handelman in 1993 [3]

While the original interest in shift and strong shift equivalence was in the case where $\mathbb{R} = \mathbb{Z}_+$, there is good reason to consider the question over more general rings, and Question 1 has motivation from both symbolic dynamics and matrix theory.

The study of question (1) is carried out in Chapter 2, where the fundamental connections between strong shift equivalence, shift equivalence, and algebraic K-theory are laid out. Let $\mathcal{R}$ be a ring. Let $\mathfrak{M}_n(\mathcal{R})$ denote the $n \times n$ matrices over $\mathcal{R}$. Using the maps $p_n\colon \mathfrak{M}_n(\mathcal{R}) \to \mathfrak{M}_{n+1}(\mathcal{R})$ defined by $M \mapsto M \oplus 1$ we let $\mathfrak{M}(\mathcal{R})$ denote the direct limit; thus a finite matrix $M$ is sent to $M_{\mathrm{st1}}$ in $\mathfrak{M}(\mathcal{R})$. Likewise we let $GL(\mathcal{R}) = \varinjlim GL_n(\mathcal{R})$ and $El(\mathcal{R}) = \varinjlim El_n(\mathcal{R})$, where $El_n(\mathcal{R})$ denotes the group of $n \times n$ elementary matrices over $\mathcal{R}$. A $\mathrm{GL}_n(\mathcal{R})$ equivalence $UMV = M'$ gives a $\mathrm{GL}_{n+1}(\mathcal{R})$ equivalence $p_n(U)p_n(M)p_n(V) = p_n(M')$, so $\mathrm{GL}(\mathcal{R})$ equivalence and $\mathrm{El}(\mathcal{R})$ equivalence of the objects $M_{\mathrm{st1}}$ is well defined. We say that two finite matrices $M$ and $M'$ are $\mathrm{GL}(\mathcal{R})$ equivalent or $\mathrm{El}(\mathcal{R})$ equivalent if the relation holds for $M_{\mathrm{st1}}$ and $(M')_{\mathrm{st1}}$, i.e. $UM_{\mathrm{st1}}V = (M')_{\mathrm{st1}}$ for $U, V \in GL(\mathcal{R})$ or $U, V \in \mathrm{El}(\mathcal{R})$. It is natural to identify $M_{\mathrm{st1}}$ with an $\mathbb{N} \times \mathbb{N}$ matrix (see Section 2.2).

For finite square matrices $A, B$ over $\mathcal{R}$, we show in Section 2.5 and Section 2.6 that

$$A \text{ and } B \text{ are SE-}\mathcal{R} \iff I - tA \text{ and } I - tB \text{ are } \mathrm{GL}(\mathcal{R}[t]) \text{ equivalent} \tag{1.1}$$

$$A \text{ and } B \text{ are SSE-}\mathcal{R} \iff I - tA \text{ and } I - tB \text{ are } \mathrm{El}(\mathcal{R}[t]) \text{ equivalent} \tag{1.2}$$

Given a ring $\mathcal{R}$ and a square matrix $M$ over $\mathcal{R}$, we define associated sets of square matrices over $\mathcal{R}$:

$$\mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(M) = \{M' : M' \text{ is } \mathrm{GL}(\mathcal{R}) \text{ equivalent to } M\}$$

$$\mathrm{Orb}_{\mathrm{EL}(\mathcal{R})}(M) = \{M' : M' \text{ is } \mathrm{El}(\mathcal{R}) \text{ equivalent to } M\}$$

Now suppose $A$ is any square matrix over $\mathcal{R}$. Define the elementary stabilizer

$$E(A, \mathcal{R}) = \{U \in \mathrm{GL}(\mathcal{R}[t])\colon U\mathrm{Orb}_{\mathrm{El}(\mathcal{R})}(I - tA) \subset \mathrm{Orb}_{\mathrm{El}(\mathcal{R})}(I - tA)\}\ .$$

The group $\mathrm{El}(\mathcal{R}) \subset E(A, \mathcal{R})$ is subgroup of the first algebraic K-group $K_1(\mathcal{R}[t])$;

there, $E(A,\mathcal{R}) \subset \mathrm{NK}_1(\mathcal{R})$ (see Section 2.2 for definitions). For a square matrix $B$ over $\mathcal{R}$, let $[B]_{SSE-\mathcal{R}}$ denote the set of matrices SSE-$\mathcal{R}$ to $B$; similarly define $[B]_{SE-\mathcal{R}}$. From (2.1) ,(2.2) and (2.3), for any square matrix $A$ over $\mathcal{R}$ we get a well-defined bijection (Theorem 2.15),

$$\begin{aligned} \mathrm{NK}_1(\mathcal{R})/E(A,\mathcal{R}) &\to \{[B]_{SSE-\mathcal{R}} \mid [A]_{SE-\mathcal{R}} = [B]_{SE-\mathcal{R}}\} \\ [I-tN] &\mapsto [A \oplus N]_{SSE-\mathcal{R}} \ . \end{aligned} \tag{1.3}$$

We do not know whether $E(A,\mathcal{R})$ can be nontrivial (Question 2.27). It is easy to check $E(A,\mathcal{R})$ is trivial if $A$ is nilpotent. We will show $E(A,\mathcal{R})$ is trivial if $A$ is SE-$\mathcal{R}$ to a matrix which is invertible or idempotent (Theorem 2.7), and in some other cases when $\mathcal{R}$ is the integral group ring of a finite abelian group (Cor. 2.9).

The technique used to show that $E(A,\mathcal{R})$ is trivial when $A$ is idempotent or invertible relies on the K-theory of non-commutative localizations. The key result for this is Theorem 2.1, showing that the map $K_1(\mathcal{R}[t]) \to K_1(\Omega_+^{-1}\mathcal{R}[t])$ is injective, where $\Omega_+^{-}\mathcal{R}[t]$ denotes the Cohn localization of $\mathcal{R}[t]$ with respect to a certain class of matrices. The proof of Theorem 2.1, in the general case, relies on the work of Neeman and Ranicki on the K-theory of non-commutative localization.

## 1.2 Finite group extensions of shifts of finite type

Chapter 3 is concerned with finite group extensions of shifts of finite type. A theorem of Livšic shows that for certain hyperbolic dynamical systems $T : X \to X$, if the restrictions of Hölder functions $f$ and $g$ to the periodic points are cohomologous as point set maps (i.e. ignoring topology), then they are Hölder cohomologous in

$(X,T)$ — i.e., $f = g + r \circ T - r$, with the transfer function $r$ being Hölder continuous. This result was generalized to nonabelian groups for shifts of finite type by Parry (see Remark 3.26) and Schmidt [4,5], and to more sophisticated systems by various authors (e.g. [4–7]).

For $(X,T)$ a mixing shift of finite type and $f : X \to G$, a suitable dynamical zeta function $\zeta_f$ encodes for all $n, g$ the number of periodic orbits of size $n$ and weight $g$. Then $\zeta_f = \zeta_g$ if and only if there is a bijection $\beta : \mathrm{Per}(X) \to \mathrm{Per}(X)$ such that $f \circ \beta$ and $g$ are cohomologous as point set maps. Parry asked, for $f : X \to G$ continuous and $G$ a finite abelian group: does the set of continuous $g : X \to G$ with $\zeta_g = \zeta_f$ contain only finitely many continuous cohomology classes?

We show in Chapter 3 that for many groups $G$ (the finite abelian groups $G$ with $NK_1(\mathbb{Z}G) \neq 0$), the answer to Parry's question is negative for many nontrivial dynamical zeta functions. Two key ingredients this are the results of Chapter 2 on the refinement of shift equivalence by strong shift equivalence for the case $\mathcal{R} = \mathbb{Z}G$, and Parry's generalization of the Williams' theory for SFTs: showing that any $G$-extension of an SFT $(X,S)$ can be presented by a square matrix $A$ over $\mathbb{Z}_+G$, and two such group extensions are isomorphic if and only if their presenting matrices are strong shift equivalent (SSE) over the positive semiring $\mathbb{Z}_+G$ of the integral group ring $\mathbb{Z}G$.

## 1.3 The Generalized Spectral Conjecture for nonnegative matrices

Chapter 4 is concerned with the following spectral conjecture for primitive matrices of Boyle & Handelman, posed in [8]:

**Spectral Conjecture:** [8] Let $\mathcal{R}$ be a subring of $\mathbb{R}$. Then $\Delta$ is the nonzero spectrum of some primitive matrix over $\mathcal{R}$ if and only if the following conditions hold:

1. $\Delta$ has a Perron value.
2. The coefficients of the polynomial $\prod_{i=1}^{k}(t-d_i)$ lie in $\mathcal{R}$.
3. If $\mathcal{R}=\mathbb{Z}$, then for all positive integers $n$, $\operatorname{tr}_n(\Delta)\geq 0$;

   if $\mathcal{R}\neq\mathbb{Z}$, then for all positive integers $n$ and $k$,

   (i) $\operatorname{tr}(\Delta^n)\geq 0$ and (ii) $\operatorname{tr}(\Delta^n)>0$ implies $\operatorname{tr}(\Delta^{nk})>0$.

Here $\Delta=(d_1,\ldots,d_k)$ is a $k$-tuple of nonzero complex numbers. $\Delta$ is the *nonzero spectrum* of a matrix $A$ if $A$ has characteristic polynomial of the form $\chi_A(t)=t^m\prod_{1\leq i\leq k}(t-d_i)$. $\Delta$ has a *Perron value* if there exists $i$ such that $d_i>|d_j|$ when $j\neq i$. The *trace* of $\Delta$ is $\operatorname{tr}(\Delta)=d_1+\cdots+d_k$. $\Delta^n$ denotes $((d_1)^n,\ldots,(d_k)^n)$, the tuple of $n$th powers; and the *nth net trace* of $\Delta$ is

$$\operatorname{tr}_n(\Delta)=\sum_{d|n}\mu(n/d)\operatorname{tr}(\Delta^d)$$

in which $\mu$ is the Möbius function ($\mu(1)=1$; $\mu(n)=(-1)^r$ if $n$ is the product of $r$ distinct primes; $\mu(n)=0$ if $n$ is divisible by the square of a prime).

The Spectral Conjecture was proved for various rings, including $\mathcal{R} = \mathbb{R}$ by Boyle and Handelman in [8]. For $\mathcal{R} = \mathbb{Z}$ it was proved by Kim, Ormes, and Roush in [9]. The Spectral Conjecture was generalized by Boyle and Handelman to include two more general versions:

**Generalized Spectral Conjecture (weak form, 1991) 1.1.** *Suppose $\mathcal{R}$ is a subring of $R$ and $A$ is a square matrix over $\mathcal{R}$ whose nonzero spectrum satisfies the three necessary conditions of the Spectral Conjecture. Then $A$ is SE over $\mathcal{R}$ to a primitive matrix.*

**Generalized Spectral Conjecture (strong form, 1993) 1.2.** *Suppose $\mathcal{R}$ is a subring of $R$ and $A$ is a square matrix over $\mathcal{R}$ whose nonzero spectrum satisfies the three necessary conditions of the Spectral Conjecture. Then $A$ is SSE over $\mathcal{R}$ to a primitive matrix.*

We note that two matrices are SE or SSE over $\mathbb{R}$ if and only if the non-nilpotent parts of their Jordan form agree. The weak form was stated in [8, p.253] and [3, p.124]. The strong form was stated in [10, Sec. 8.4]), where the authors state that they were not aware whether the conjectures were equivalent (not knowing if shift equivalence over a ring implies strong shift equivalence over it). Following the results in Section 1, (see Theorem 4.5), we know now that the strong form of the Generalized Spectral Conjecture was not a vacuous generalization: there are subrings of $\mathbb{R}$ over which SE does not imply SSE (see Chapter 5). The results of Chapter 1 also provide enough structure that we can prove Theorem 4.1, which shows that the two forms of the Generalized Spectral Conjecture are equivalent.

This follows from the following Theorem, proved in Chapter 4.

**Theorem 1.3.** *Suppose $\mathcal{R}$ is a dense subring of $\mathcal{R}$, $A$ is a primitive matrix over $\mathcal{R}$ and $B$ is a matrix over $\mathcal{R}$ which is shift equivalent over $\mathcal{R}$ to $A$.*

*Then $B$ is strong shift equivalent over $\mathcal{R}$ to a primitive matrix.*

## 1.4 Explicit Examples in $NK_1(\mathcal{R})$

Chapter 5 contains the computation of some explicit non-zero classes in the algebraic K-group $NK_1(\mathcal{R})$ for certain rings $\mathcal{R}$. The computations were directly motivated by the results of Chapter 2, where the group $NK_1(\mathcal{R})$ arises in connection with the refinement of shift equivalence over $\mathcal{R}$ by strong shift equivalence over $\mathcal{R}$. The rings considered in this chapter were motivated by the content in Chapters 3 and 4.

## 1.5 Isolating zero-dimensional dynamics on manifolds

Chapter 6 is unrelated to the previous chapters. For a homeomorphism $f : X \to X$ of a compact metric space $X$, a compact set $I$ which is invariant under $f$ is *isolated* if there exists a neighborhood $U$ of $I$ such that $I = \cap \lim_{f^n}(U)$. Isolated invariant sets lie at the heart of Conley index theory, a vast toolset that has proven successful in analyzing complicated systems, and has undergone significant development since its inception (see [11] for a brief survey). In the case $f$ is a hyperbolic map, isolated invariant sets are more frequently referred to as *locally maximal* sets, and in the smooth hyperbolic setting significantly more is known regarding their structure (see [12, Section 18.4], [13]). A fundamental example of such an isolated invariant set is given by the Smale horseshoe map $S : S^2 \to S^2$, for which there exists an isolated invariant set on which $S$ is conjugate to the full 2-shift.

**Question** $I$**.** We consider the case of a homeomorphism $f : M \to M$ of a compact manifold $M$, and investigate the following question: if $I \subset M$ is compact, zero-dimensional, and isolated by $f$, what can the dynamics of $(I, f|_I)$ be?

We show that, in the case $M$ is of dimension 3 or greater, any compact zero-dimensional system can arise as the isolated invariant set for a homeomorphism of $M$. More precisely, we prove the following.

**Theorem 1.4** (Theorem 6.2 in Chapter 6)**.** *Let $M$ be a compact 2-manifold and let $g : X \to X$ be a homeomorphism of a compact zero-dimensional space $X$. For any $n \geq 1$, there exists a homeomorphism $f : M \times \mathbb{T}^n$ for which $f$ contains an isolated invariant set $I$ such that $f|_I$ is topologically conjugate to the system $g : X \to X$.*

Question $I$ in the case $M$ is dimension two is significantly more delicate, and the remainder of Chapter 5 is devoted to this case. Let us say a compact zero-dimensional system $(X, g)$ is *(orientably) isolatable in dimension two* if there exists a homeomorphism $f : M \to M$ of a compact two-manifold $M$ containing an invariant set on which $f$ is topologically conjugate to the system $(X, g)$. We prove in Section 5.1 that the question of whether a given compact zero-dimensional system $(X, f)$ is orientably isolatable in dimension two depends only on the flow equivalence class of $(X, f)$. Sections 6.5-6.8 provide examples of homeomorphisms of two-manifolds which contain isolated invariant sets exhibiting a wide range of behavior. Section

6.10 presents algebraic obstructions for a compact zero-dimensional system $(X, g)$ to be isolatable in dimension two. These obstructions can be presented in terms of only the dynamics of the system $(X, g)$. Finally, Section 6.11 proves that if a homeomorphism $f : M \to M$ of compact two-manifold contains an invariant set $X$ on which $f$ is conjugate to an odometer, then $X$ must be the limit of $f$-periodic points in $M$.

# Chapter 2: Strong shift equivalence and algebraic K-theory

## 2.1 Introduction

Let $\mathcal{R}$ (always assumed to contain 0 and 1) be a subset of a ring. Let $A, B$ be square matrices over $\mathcal{R}$ (not necessarily of equal size). Matrices $A$ and $B$ over $\mathcal{R}$ are *elementary strong shift equivalent* over $\mathcal{R}$ (ESSE-$\mathcal{R}$) if there exist matrices $U, V$ over $\mathcal{R}$ such that $A = UV$ and $B = VU$. $A$ and $B$ are *strong shift equivalent* over $\mathcal{R}$ (SSE-$\mathcal{R}$) if they are connected by a chain of elementary strong shift equivalences. $A$ and $B$ are *shift equivalent* over $\mathcal{R}$ (SE-$\mathcal{R}$) if there exist matrices $U, V$ over $\mathcal{R}$ and $\ell$ in $\mathbb{N}$ such that the following hold:

$$\begin{aligned} A^\ell &= UV \qquad & B^\ell &= VU \\ AU &= UB \qquad & VA &= BV \quad . \end{aligned}$$

If $A, B$ are SSE-$\mathcal{R}$, then they are SE-$\mathcal{R}$.

For symbolic dynamics, these are central relations, introduced by Williams [1, 14]; they may be familiar from other settings. For example, idempotent matrices $p, q$ over a unital $C^*$-algebra $\mathcal{A}$ are Murray-von Neumann equivalent if and only if $p$ and $q$ are SSE-$\mathcal{R}$ (this can be deduced from [15, Lemma A.4.4]). We give background and motivation in Section 2.2. Briefly: shift equivalence is very useful for symbolic dynamics and reasonably tractable, with several algebraic characterizations when $\mathcal{R}$ is a ring (see Theorem 2.12). Strong shift equivalence is a more fundamental and mysterious relation.

There is an obvious basic question: assuming $\mathcal{R}$ is a ring, does SE $-\mathcal{R}$ imply SSE $-\mathcal{R}$? The answer was shown to be yes for $\mathcal{R} = \mathbb{Z}$ (see Williams' proof in [16] on his work from the 70s); for $\mathcal{R}$ a principal ideal domain (Effros, 1981, [2]); and for $\mathcal{R}$ a Dedekind domain (Boyle-Handelman, 1993 [3]). There were no counterexamples, and no results after [3]. In his 1999 Bulletin AMS survey, Wagoner formally posed the "Algebraic Shift Equivalence Problem" [17, Problem 2.14]: for what rings $\Lambda$ does SE over $\Lambda$ imply SSE over $\Lambda$? We will show that for $\mathcal{R}$ a ring, in a given SE $-\mathcal{R}$ class the refinement of SE $-\mathcal{R}$ by SSE $-\mathcal{R}$ is captured exactly by a certain quotient group of the algebraic K-theory group $\mathrm{NK}_1(\mathcal{R})$.

From here, let $\mathcal{R}$ be a ring, and $\mathfrak{M}_n(\mathcal{R})$ the $n \times n$ matrices over $\mathcal{R}$. With the maps $p_n \colon \mathfrak{M}_n(\mathcal{R}) \to \mathfrak{M}_{n+1}(\mathcal{R})$ defined by $M \mapsto M \oplus 1$, we form a direct limit of sets $\mathfrak{M}(\mathcal{R})$, with a finite matrix $M$ sent to $M_{\mathrm{st1}}$ in $\mathfrak{M}(\mathcal{R})$. The maps $p_n$ are the maps which construct $\mathrm{GL}(\mathcal{R})$ and the elementary group $\mathrm{El}(\mathcal{R})$ as direct limits. A $\mathrm{GL}_n(\mathcal{R})$ equivalence $UMV = M'$ gives a $\mathrm{GL}_{n+1}(\mathcal{R})$ equivalence $p_n(U)p_n(M)p_n(V) = p_n(M')$, so $\mathrm{GL}(\mathcal{R})$ equivalence and $\mathrm{El}(\mathcal{R})$ equivalence of the objects $M_{\mathrm{st1}}$ is well defined. When we say that two finite matrices $M$ and $M'$ are $\mathrm{GL}(\mathcal{R})$ equivalent or $\mathrm{El}(\mathcal{R})$ equivalent, we mean that the relation holds for $M_{\mathrm{st1}}$ and $(M')_{\mathrm{st1}}$, i.e. $UM_{\mathrm{st1}}V = (M')_{\mathrm{st1}}$ for $U, V \in GL(\mathcal{R})$ or $U, V \in \mathrm{El}(\mathcal{R})$. It is natural to identify $M_{\mathrm{st1}}$ with an $\mathbb{N} \times \mathbb{N}$ matrix (see Sec. 2.2).

For finite square matrices $A, B$ over $\mathcal{R}$, we will show

$$A \text{ and } B \text{ are SE-}\mathcal{R} \iff I - tA \text{ and } I - tB \text{ are } \mathrm{GL}(\mathcal{R}[t]) \text{ equivalent} \qquad (2.1)$$

$$A \text{ and } B \text{ are SSE-}\mathcal{R} \iff I - tA \text{ and } I - tB \text{ are } \mathrm{El}(\mathcal{R}[t]) \text{ equivalent} \qquad (2.2)$$

The proof of (2.1) in Section 2.5 uses an old stabilization result of Fitting. In Section 2.6, (2.2) is proved. The formulation of the correspondence in Theorem 4.6 as induced by a map $I - A \mapsto \mathcal{A}^{\square}$ is simple and natural. The matrix arguments of the proof, however, are nonstandard for $K$-theory, and a $K$-theorist may find the details barbaric: nonfunctorial, complicated and (worst of all?) bereft of exact sequences. For better and for worse, this is the proof we have.

Given a ring $\mathcal{R}$ and a square matrix $M$ over $\mathcal{R}$, we define associated sets of square matrices over $\mathcal{R}$:

$$\mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(M) = \{M' : M' \text{ is } \mathrm{GL}(\mathcal{R}) \text{ equivalent to } M\}$$

$$\mathrm{Orb}_{\mathrm{EL}(\mathcal{R})}(M) = \{M' : M' \text{ is } \mathrm{El}(\mathcal{R}) \text{ equivalent to } M\}$$

Now suppose $A$ is any square matrix over $\mathcal{R}$. Then $\mathrm{Orb}_{\mathrm{GL}(\mathcal{R}[t])}(I - tA)$ is a disjoint union of the sets $\mathrm{Orb}_{\mathrm{EL}(\mathcal{R}[t])}(I - tB)$ such that $I - tB$ is $\mathrm{GL}(\mathcal{R}[t])$ equivalent to $I - tA$ and $B$ has entries in $\mathcal{R}$. Define the elementary stabilizer

$$E(A, \mathcal{R}) = \{U \in \mathrm{GL}(\mathcal{R}[t]) \colon U\mathrm{Orb}_{\mathrm{El}(\mathcal{R})}(I - tA) \subset \mathrm{Orb}_{\mathrm{El}(\mathcal{R})}(I - tA)\}\ .$$

Because $\mathrm{El}(\mathcal{R}) \subset E(A, \mathcal{R})$, we may also regard $E(A, \mathcal{R})$ as a subgroup of $K_1(\mathcal{R}[t])$; there, $E(A, \mathcal{R}) \subset \mathrm{NK}_1(\mathcal{R})$. (We will recall definitions in Section 2.2.)

We will show that there is a bijection

$$\begin{aligned} \mathrm{NK}_1(\mathcal{R})/E(A,\mathcal{R}) &\to \{\mathrm{Orb}_{\mathrm{EL}(\mathcal{R}[t])}(I-tB) : I-tB \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R}[t])}(I-tA)\} \\ [I-tN] &\mapsto \mathrm{Orb}_{\mathrm{EL}(\mathcal{R}[t])}(I-t(A\oplus N))\ . \end{aligned} \tag{2.3}$$

In (2.3), $B$ is a square matrix over $\mathcal{R}$; $N$ is a nilpotent matrix over $\mathcal{R}$; and $[I-tN]$ is the class in $\mathrm{NK}_1(\mathcal{R})$ containing $I-tN$.

For a square matrix $B$ over $\mathcal{R}$, let $[B]_{SSE-\mathcal{R}}$ denote the set of matrices SSE-$\mathcal{R}$ to $B$; similarly define $[B]_{SE-\mathcal{R}}$. From (2.1) ,(2.2) and (2.3), for any square matrix $A$ over $\mathcal{R}$ we get a well-defined bijection (Theorem 2.15),

$$\begin{aligned} \mathrm{NK}_1(\mathcal{R})/E(A,\mathcal{R}) &\to \{[B]_{SSE-\mathcal{R}} \mid [A]_{SE-\mathcal{R}} = [B]_{SE-\mathcal{R}}\} \\ [I-tN] &\mapsto [A\oplus N]_{SSE-\mathcal{R}}\ . \end{aligned} \tag{2.4}$$

We do not know whether $E(A,\mathcal{R})$ can be nontrivial (Question 2.27). It is easy to check $E(A,\mathcal{R})$ is trivial if $A$ is nilpotent. There are rings with nontrivial $\mathrm{NK}_1(\mathcal{R})$ such that $E(A,\mathcal{R})$ is trivial for every $A$ (Remark 2.25). We will show $E(A,\mathcal{R})$ is trivial if $A$ is SE-$\mathcal{R}$ to a matrix which is invertible or idempotent (Theorem 2.7), and in some other cases when $\mathcal{R}$ is the integral group ring of a finite abelian group (Cor. 2.9).

The key to the triviality of $E(A,\mathcal{R})$ for invertible or idempotent $A$ (important for applications) is Theorem 2.1, which shows that the map $K_1(\mathcal{R}[t]) \to K_1(\Omega_+^{-1}\mathcal{R}[t])$ induced by a certain Cohn localization $\mathcal{R}[t] \to \Omega_+^{-1}\mathcal{R}[t]$ is injective. In the case $\mathcal{R}$ is commutative, this can be handled with a standard localization exact sequence. But for the generality of all rings $\mathcal{R}$, the proof depends on the work of Neeman and Ranicki on the $K$-theory of noncommutative localization. For general $\mathcal{R}$, they extended a localization finite exact sequence of Schofield by a single term (see Theorem 2.3). We need that extra term to prove Theorem 2.1. At the end of Section 2.4, we provide some context for the statement and proof of Theorem 2.7. In Section 2.7, we note that for nilpotent matrices $N, N'$ over $\mathcal{R}$, $[N] = [N']$ in $\mathrm{Nil}_0(\mathcal{R})$ if and only if $N$ and $N'$ are SSE-$\mathcal{R}$, and add a remark about the standard isomorphism $\mathrm{Nil}_0(\mathcal{R}) \to \mathrm{NK}_1(\mathcal{R})$.

## 2.2 Background and applications

In this section, we give basic definitions we need for $K$-theory, shift equivalence and strong shift equivalence. Then we give a little background from symbolic dynamics (not needed for proofs), and summarize motivations and applications.

**Notational convention 2.5.** *Let $M_{\mathrm{st1}}$ be defined as in the introduction from a finite square matrix $M$. We regard $M_{\mathrm{st1}}$ as an $\mathbb{N} \times \mathbb{N}$ matrix which has $M$ as its upper left corner and is otherwise equal to the identity matrix. In the set of $\mathbb{N} \times \mathbb{N}$ matrices, $I$ denotes the infinite identity matrix. Thus $\mathfrak{M}(\mathcal{R})$ becomes the set of all $\mathbb{N} \times \mathbb{N}$ matrices over $\mathcal{R}$ equal to $I$ outside finitely many entries. To avoid a heavier notation, we sometimes suppress the subscript $_{\mathrm{st1}}$. For example, if $M$ is a finite square matrix and $U$ in $\mathrm{GL}(\mathcal{R})$, then $UM$ means $UM_{\mathrm{st1}}$. When we say finite square matrices $M, M'$ are $\mathrm{GL}(\mathcal{R})$ equivalent, we mean there are $U, V$ in $\mathrm{GL}(\mathcal{R})$ such that $UM_{\mathrm{st1}}V = (M')_{\mathrm{st1}}$.*

*Remark* 2.6. If in the introduction for $p_n$ we used $M \mapsto M \oplus 0$ rather than $M \mapsto$

$M\oplus 1$, we would produce a more standard stable version of $M$, which we denote $M_{\mathrm{st0}}$. Consistent with the matrix interpretation of $M_{\mathrm{st1}}$, we regard $M_{\mathrm{st0}}$ as an $\mathbb{N}\times\mathbb{N}$ matrix which has upper left corner $M$ and has other entries zero. With this interpretation, $(I_n - A)_{\mathrm{st1}} = I - A_{\mathrm{st0}}$ .

**Some basic K-theory.** Throughout this paper, a ring means a ring with unit. Unless mentioned otherwise, for $\mathcal{R}$ a ring, an $\mathcal{R}$-module $M$ is a right $\mathcal{R}$-module ($r : m \mapsto mr$), and matrix multiplication of vectors is multiplication of column vectors. Everything in the paper would remain true if instead we used left $\mathcal{R}$ modules and muitiplication of row vectors.

We briefly review some definitions and notation. We recommend the books [18, 19] for an introduction to algebraic K-theory.

Let $\mathcal{R}$ be a ring. The group $K_1(\mathcal{R})$ is defined by $K_1(\mathcal{R}) = \mathrm{GL}(\mathcal{R})/\,\mathrm{El}(\mathcal{R})$, where $\mathrm{GL}(\mathcal{R}) = \varinjlim \mathrm{GL}_n(\mathcal{R})$ and $\mathrm{El}(\mathcal{R}) = \varinjlim \mathrm{El}_n(\mathcal{R})$, with $\mathrm{El}_n(\mathcal{R})$ the group generated by basic elementary matrices of size $n$ (those equal to $I$ except possibly in a single offdiagonal entry). If $\mathcal{R}$ is commutative, then $\mathrm{El}(\mathcal{R}) \subset \mathrm{SL}(\mathcal{R}) := \varinjlim \mathrm{SL}_n(\mathcal{R})$, and $\mathrm{SK}_1(\mathcal{R})$ denotes $\{[M] \in K_1(\mathcal{R}) : \det M = 1\}$. As above, we use $\mathbb{N}\times\mathbb{N}$ matrices as a notation for these direct limits. The group $NK_1(\mathcal{R})$ is the kernel of the homomorphism $K_1(\mathcal{R}[t]) \to K_1(\mathcal{R})$ induced by the ring homomorphism $\mathcal{R}[t] \overset{t\to 0}{\to} \mathcal{R}$. The exact sequence $0 \to t\mathcal{R}[t] \to \mathcal{R}[t] \overset{t\to 0}{\to} \mathcal{R} \to 0$ is split on the right, giving a decomposition $K_1(\mathcal{R}[t]) \cong NK_1(\mathcal{R}) \oplus K_1(\mathcal{R})$.

For a category $\mathcal{P}$ with exact sequences and small skeleton $\mathcal{P}_0$, $K_0(\mathcal{P})$ is defined to be the free abelian group on $Obj(\mathcal{P}_0)$, modulo the relations:

(1) $[P_1] = [P_2]$ if $P_1$ and $P_2$ are isomorphic in $\mathcal{P}$.

(2) $[P] = [P_1] + [P_2]$ if there is a short exact sequence in $\mathcal{P}$

$$0 \to P_1 \to P \to P_2 \to 0$$

For a ring $\mathcal{R}$, the nil category $\mathbf{Nil(\mathcal{R})}$ is the exact category whose objects are pairs $(P, f)$, where $P$ is an object in $\mathbf{Proj(\mathcal{R})}$, the category of finitely generated projective $\mathcal{R}$-modules, and $f$ is a nilpotent endomorphism of $P$. A morphism $h\colon (P, f) \to (Q, g)$ in $\mathbf{Nil(\mathcal{R})}$ is a morphism $h\colon P \to Q$ in $\mathbf{Proj(\mathcal{R})}$ such that

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "f"] & Q \arrow[d, "g"] \\ P \arrow[r, "h"] & Q \end{tikzcd}$$

commutes. There is a split surjective functor $\mathbf{Nil}(\mathcal{R})\to\mathbf{Proj}\ \mathcal{R}$ defined by sending $(P, f)$ to $P$, and we let $\mathrm{Nil}_0(\mathcal{R})$ denote the kernel of $K_0(\mathbf{Nil(\mathcal{R})})\to K_0(\mathcal{R})$, giving a decomposition $K_0(\mathbf{Nil(\mathcal{R})}) = K_0(\mathcal{R}) \oplus \mathrm{Nil}_0(\mathcal{R})$.

Every element of $\mathrm{NK}_1(\mathcal{R})$ contains a matrix of the form $I - tN$, with $N$ a nilpotent matrix with entries in $\mathcal{R}$. It is a classic result that the map $[I-tN] \to [N]$ defines an isomorphism $\mathrm{NK}_1(\mathcal{R}) \to \mathrm{Nil}_0(\mathcal{R})$. A theorem of Farrell [20] shows that when $\mathrm{NK}_1(\mathcal{R}) \neq 0$, $\mathrm{NK}_1(\mathcal{R})$ is not finitely generated as a group. If $G$ is a finite group of order $n$, then $\mathrm{NK}_1(\mathbb{Z}G)$ is trivial if $n$ is square-free [21], but in general may not vanish [22].

To appreciate that $NK_1(\mathcal{R})$ is often trivial, recall that a (left) Noetherian ring is regular if every finitely generated (left) $\mathcal{R}$-module $M$ has a finite-type projective resolution, i.e. there exists an exact sequence

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

with $P_i$ projective for all $i$. These Noetherian regular rings form a large class, containing rings of finite global dimension (fields, principal ideal domains, Dedekind domains ...). If $\mathcal{R}$ is regular, then the polynomial ring $\mathcal{R}[x_1, \ldots, x_n]$ is regular. If $\mathcal{R}$ is a Noetherian regular ring, then $NK_1(\mathcal{R})$ is trivial.

**Cohn Localization.** Cohn localization is a fundamental tool for the study of noncommutative rings.

Let $\Sigma$ be a collection of matrices over a ring $\mathcal{R}$, $\Sigma = \{A_i\}$. The Cohn localization of $\mathcal{R}$ with respect to $\Sigma$ consists of a ring (denoted $\Sigma^{-1}\mathcal{R}$) with a ring homomorphism $\phi\colon \mathcal{R} \to \Sigma^{-1}\mathcal{R}$ satisfying two properties:

1. For every matrix $A$ in $\Sigma$, $\phi(A)$ is invertible in $\Sigma^{-1}\mathcal{R}$.

2. If $\gamma\colon \mathcal{R} \to S$ is any other ring homomorphism such that $\gamma(A)$ is invertible over $S$ for all $A \in \Sigma$, then there is a (unique) ring homomorphism $\delta\colon \Sigma^{-1}\mathcal{R} \to S$ such that $\gamma = \phi \circ \delta$.

The ring $\Sigma^{-1}\mathcal{R}$ is thus a universal $\Sigma$-inverting ring. With the usual nontriviality assumption for a ring, $0 \neq 1$, there might be no ring over which the matrices in $\Sigma$ become invertible. Therefore, so that $\Sigma^{-1}\mathcal{R}$ is always defined, the degenerate possibility $\Sigma^{-1}\mathcal{R} = \{0\}$ is allowed. Then $\Sigma^{-1}\mathcal{R}$ exists and is essentially unique (see [23] or [24]).

The Cohn localization can also be constructed given a collection of morphisms between finitely generated projective $\mathcal{R}$-modules in an analogous fashion. Given such a collection $\Sigma$, call a ring morphism $\mathcal{R} \to \mathcal{S}$ $\Sigma$-inverting if $\sigma \otimes 1 \colon P \otimes_{\mathcal{R}} \mathcal{S} \to Q \otimes_{\mathcal{R}} \mathcal{S}$ is an $\mathcal{S}$-module isomorphism for every $\sigma \colon P \to Q$ in $\Sigma$. Then the noncommutative localization is a ring $\Sigma^{-1}\mathcal{R}$ with a $\Sigma$-inverting map $\mathcal{R} \to \Sigma^{-1}\mathcal{R}$ such that $\Sigma^{-1}\mathcal{R}$ is universal with respect to $\Sigma$-inverting maps, analogous to (2) above.

More details regarding the general construction of $\Sigma^{-1}\mathcal{R}$ may be found in 7.2 of [24].

Given $\mathcal{R}$, define $\Omega_+$ to be the collection of $\mathcal{R}[t]$-module homomorphisms satisfying the following:

1. Each $f \in \Omega_+$ is an $\mathcal{R}[t]$-module homomorphism $f \colon P \to Q$ between some finitely generated $\mathcal{R}[t]$-modules $P, Q$.

2. For every $f \in \Omega_+$, $f$ is injective, and $\operatorname{coker}(f)$ is a finitely generated projective $\mathcal{R}$-module.

Following [25], we refer to $\Omega_+$ as the set of Fredholm homomorphisms. The localization $\Omega_+^{-1}\mathcal{R}[t]$ has the property that the map $\mathcal{R}[t] \to \Omega_+^{-1}\mathcal{R}[t]$ is injective [25, Prop. 10.7].

One can alternatively construct the Fredholm localization using matrices. Let $\Omega_+^{\mathrm{mat}}$ denote the set of matrices $A$ over $\mathcal{R}[t]$ such that (with $A$ $m \times n$) the induced map on free $\mathcal{R}[t]$-modules $\mathcal{R}[t]^n \xrightarrow{A} \mathcal{R}[t]^m$ is injective and $\operatorname{coker}(A)$ is a finitely generated projective $\mathcal{R}$-module. We refer to $\Omega_+^{\mathrm{mat}}$ as the set of Fredholm matrices. That the localizations $\Omega_+^{-1}\mathcal{R}[t]$ and $(\Omega_+^{\mathrm{mat}})^{-1}\mathcal{R}[t]$ coincide is easy to check. We may occasionally abuse notation and write $\Omega_+$ in place of $\Omega_+^{\mathrm{mat}}$ when it is clear that matrices are being considered.

An alternative construction of $\Omega_+^{-1}\mathcal{R}[t]$ may be described as follows. Let $\Omega_M$ denote the set of monic matrices over $\mathcal{R}[t]$, i.e. the square matrices $A = \sum_{i=0}^{d} A_i t^i$ with the $A_i$ matrices over $\mathcal{R}$ such that $A_d$ is the identity matrix. Note that $\Omega_M \subset \Omega_+$. In fact, the two localizations coincide [25, Prop. 10.7]: $\Omega_+^{-1}\mathcal{R}[t] = \Omega_M^{-1}\mathcal{R}[t]$.

**Shift equivalence** Two square matrices $A, B$ over $\mathcal{R}$ are called shift equivalent over $\mathcal{R}$ (SE-$\mathcal{R}$) if there exists a positive integer $l$ (the lag) and matrices $R, S$ over $\mathcal{R}$ such that

$$RS = A^l, SR = B^l, RB = AR, BS = SA.$$

While shift equivalence is an equivalence relation, lag one shift equivalence is not. The transitive closure of lag one shift equivalence is called strong shift equivalence, so two square matrices $A, B$ over $\mathcal{R}$ are strong shift equivalent over $\mathcal{R}$ (SSE-$\mathcal{R}$) if there is a chain of lag one shift equivalences between them.

**Strong shift equivalence** Let $\mathcal{R}$ be a ring. The nature of SSE-$\mathcal{R}$ as a kind of stabilized version of similarity over $\mathcal{R}$ is shown by the following characterization from [26]. The relation SSE-$\mathcal{R}$ is generated by two relations:

(1) Similarity over $\mathcal{R}$: $A = U^{-1}BU$.

(2) "Zero extensions":

$$\begin{pmatrix} A & U \\ 0 & 0 \end{pmatrix} \sim A \sim \begin{pmatrix} A & 0 \\ U & 0 \end{pmatrix}$$

Similarity over $\mathcal{R}$ implies SSE-$\mathcal{R}$, since $A = U^{-1}BU$ gives $A = VU$, $B = UV$ with

$V = U^{-1}B$. Each type of zero extension respects SSE-$\mathcal{R}$, because

$$A = \begin{pmatrix} A & U \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} , \qquad \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{pmatrix} A & U \end{pmatrix} = \begin{pmatrix} A & U \\ 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} A \\ U \end{pmatrix} , \qquad \begin{pmatrix} A \\ U \end{pmatrix} \begin{pmatrix} I & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ U & 0 \end{pmatrix} .$$

Conversely, given $A = UV,\ B = VU$ we have a similarity:

$$\begin{pmatrix} I & 0 \\ V & I \end{pmatrix} \begin{pmatrix} A & U \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & U \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ V & I \end{pmatrix} \tag{2.7}$$

**Antecedents.** The connection between $\mathrm{Nil}_0(\mathcal{R})$ and SSE $-\mathcal{R}$ grew for us out of the "positive K-theory" [27, 28] approach to classification problems in symbolic dynamics. That approach grew out of earlier work, especially [29–31], and Wagoner's background in algebraic K-theory. Some classification problems in symbolic dynamics can be presented, for a suitable ordered ring $\mathcal{R}$, as the problem of classifying square matrices $A, B$ over $\mathcal{R}$ up to SSE $-\mathcal{R}_+$. In the most important example, for the classification of shifts of finite type, Williams used $\mathcal{R} = \mathbb{Z}_+$ [1]. For the classification of group extensions of shifts of finite type by a finite group $G$ for example, Parry used $\mathcal{R} = \mathbb{Z}_+G$ [32, 33]. For a group ring $\mathcal{R} = \mathbb{Z}G$, the relation SSE $-\mathbb{Z}_+G$ of $A$ and $B$ is equivalent to "positive" equivalence of the matrices $I - tA$ and $I - tB$ [28, Theorem 7.2]. Here a positive equivalence is a certain type of $\mathrm{El}(\mathbb{Z}G[t])$ equivalence $U(I - tA)V = I - tB$ (see [27, 28, 33] for definitions and explanation). This by analogy raises the question for rings answered by (2.2).

The elementary stabilizer as a subgroup of $K_1(\mathcal{R})$ appeared in a related context in [32] (see Remark 2.31).

**Motivation and applications.** The results in this paper have been used to answer (in the negative) a question of Parry [34, Sec. 4.4] about a possible extension of Livšic theory to finite group extensions of shifts of finite type, and have significantly clarified the structure of their algebraic invariants [33]. They have also been used to show that two old conjectures about the algebraic structure of nonnegative matrices are equivalent [35].

The papers [33, 35] appealed to more general statements than we can now prove, as explained below in Remark 2.28. However, the arguments of [35] go through unchanged, with appropriate reference to Theorem 2.13 in place of [35, Theorem 2.1]. In [33], after replacing Theorem 2.2(2) with a reference to Theorem 2.13 below, the theorems and proofs remain correct, with one amendment: in Theorem 6.4 of [33], there should be added the assumption that the elementary stabilizer $E(A, \mathcal{R})$ (Defn. 2.23) is trivial (which by Theorem 2.13 holds in many cases). The revised Theorem 6.4 still provides many cases in which the answer to Parry's question is decisively no.

In [36], a three part program for understanding SSE for positive real matrices was proposed. One part, understanding the refinement of SSE by SE for subrings of $\mathbb{R}$, is addressed by the current paper.

One "application" of a result describing the refinement of SE by SSE is that one acquires constraints on what proofs might possibly work. For example, the main result of [36] had a hypothesis of SSE (not SE) of two matrices over a subring of $\mathbb{R}$. We now know that hypothesis is not an artifact of the proof.

The classification problem for shifts of finite type is a central open problem for symbolic dynamics. Wagoner used $K_2$ of the dual numbers as an ingredient for producing a counterexample to Williams' conjecture that SE-$\mathbb{Z}_+$ implies SSE-$\mathbb{Z}_+$, and suggested further possible connection between the classification problem and algebraic K-theory [37, 38]. The current paper is, we hope, a step toward understanding that connection.

## 2.3 $K_1(\mathcal{R}[t]) \to K_1(\Omega_+^{-1}\mathcal{R}[t])$ is injective

The main purpose of this section is to prove Theorem 2.1, which we need to prove Theorem 2.6.

**Theorem 2.1.** *Let $\Omega_+$ denote the set of Fredholm homomorphisms of finitely generated projective modules over $\mathcal{R}[t]$. Then the natural map*

$$K_1(\mathcal{R}[t]) \to K_1(\Omega_+^{-1}\mathcal{R}[t])$$

*induced by $\mathcal{R}[t] \to \Omega_+^{-1}\mathcal{R}[t]$ is injective.*

The proof of Theorem 2.1 for general $\mathcal{R}$ requires us to delve into the proofs behind the Neeman and Ranicki results on the $K$-theory of Cohn localizations. Before going to that more difficult work, we'll give the (shorter) proof for the case that $\mathcal{R}$ is commutative. The proof for this case uses the standard K-theory localization exact sequence (2.9) with claims appealing to standard references. After that, we will be better positioned to understand (and appreciate) how the work of Neeman and Ranicki fits in. We provide more explanation and reference than experts might need, in an effort to make the material more widely accessible and easily checked.

### The Commutative Case

In this subsection, $\mathcal{R}$ is assumed to be commutative.

**Definition 2.8.** For a ring $\mathcal{R}$, we consider the following exact categories:

1. $\mathcal{H}_1(\mathcal{R})$ is the exact category whose objects are $\mathcal{R}$-modules which have a resolution of length $\leq 1$ by finitely generated projective $\mathcal{R}$-modules, and whose morphisms are the $\mathcal{R}$-module homomorphisms between them.

2. Given a multiplicatively closed set $S \subset \mathcal{R}$ of non-zero divisors, $\mathcal{H}_{1,S}(\mathcal{R})$ denotes the full subcategory of $\mathcal{H}_1(\mathcal{R})$ whose objects are the objects of $\mathcal{H}_1(\mathcal{R})$ which are $S$-torsion modules (i.e. $sM = 0$ for some $s \in S$).

Our use of the term exact category matches the standard one, as in [19, Definition II.7.0]. The notation $\mathcal{H}_1(\mathcal{R})$ was chosen to match Weibel's K-Book [19, Definition II.7.7]. It follows from the Resolution Theorem [19, V.3.1] that the inclusion of Proj$\mathcal{R}$ into $\mathcal{H}_1(\mathcal{R})$ induces an isomorphism $\rho\colon K_1(\mathcal{R}) \to K_1(\mathcal{H}_1(\mathcal{R}))$. The category $\mathcal{H}_{1,S}(\mathcal{R})$ appears in the standard long exact sequence [19, V.7.1]

$$\cdots \to K_n(\mathcal{H}_{1,S}(\mathcal{R})) \to K_n(\mathcal{R}) \to K_n(S^{-1}\mathcal{R}) \to \cdots \tag{2.9}$$

which holds for the localization of a commutative ring $\mathcal{R}$ at a multiplicatively closed set $S$ of central non-zero divisors.

Let $S_+$ denote the collection of monic polynomials in $\mathcal{R}[t]$, i.e. polynomials of the form $p(t) = \sum_{i=0}^{n} a_i t^i$ with $a_n = 1$. The set $S_+$ is a multiplicatively closed set of non-zero divisors. Replacing $\mathcal{R}$ and $S$ in (2.9) with $\mathcal{R}[t]$ and $S_+$, we get the exact sequence

$$\cdots \to K_n(\mathcal{H}_{1,S_+}(\mathcal{R}[t])) \to K_n(\mathcal{R}[t]) \to K_n(S_+^{-1}\mathcal{R}[t]) \to \cdots \tag{2.10}$$

To prove Theorem 2.1 for $\mathcal{R}$ commutative, it is now sufficient to show that the map $\alpha\colon K_1(\mathcal{H}_{1,S_+}(\mathcal{R}[t])) \to K_1(\mathcal{R}[t])$ in (2.10) is the zero map. This map factors through the map induced by the inclusion functor (see the proof of [19, V.7.1]) $j\colon \mathcal{H}_{1,S_+}(\mathcal{R}[t]) \to \mathcal{H}_1(\mathcal{R}[t])$, giving a diagram

$$\begin{tikzcd} K_1(\mathcal{H}_{1,S_+}(\mathcal{R}[t])) \arrow[r, "K_1(j)"] \arrow[dr, "\alpha"] & K_1(\mathcal{H}_1(\mathcal{R}[t])) \arrow[d, "\rho^{-1}"] \\ & K_1(\mathcal{R}[t]) \end{tikzcd}$$

in which the vertical map is the inverse to the isomorphism $K_1(\mathrm{Proj}\mathcal{R}[t]) \to K_1(\mathcal{H}_1(\mathcal{R}[t]))$ given by the Resolution Theorem. It suffices then to show the map

$$K_1(j)\colon K_1(\mathcal{H}_{1,S_+}(\mathcal{R}[t])) \to K_1(\mathcal{H}_1(\mathcal{R}[t]))$$

is the zero map.

For $M$ in $\mathcal{H}_{1,S_+}(\mathcal{R}[t])$, define $\eta(M) = M \otimes_{\mathcal{R}} \mathcal{R}[t]$. The right $\mathcal{R}[t]$-module $\eta(M)$ carries no memory of the original action of $t$ on $M$; as an $\mathcal{R}$-module, it is isomorphic to a direct sum of countably many copies of $M$. A well known argument [39, p. 441] shows that every object $M$ in $\mathcal{H}_{1,S_+}(\mathcal{R}[t])$ is finitely generated projective as an $\mathcal{R}$-module. For $M$ in $\mathcal{H}_{1,S_+}(\mathcal{R}[t])$, it follows that $\eta(M)$ is a finitely generated projective $\mathcal{R}[t]$-module, and hence lies in $\mathcal{H}_1(\mathcal{R}[t])$. Let $\eta$ also denote the functor $\mathcal{H}_{1,S_+}(\mathcal{R}[t]) \to \mathcal{H}_1(\mathcal{R}[t])$ which is $M \mapsto \eta(M)$ on objects and $f \mapsto f \otimes_{\mathcal{R}} \mathrm{id}$ on morphisms. The functor $\eta$ is exact, since $\mathcal{R}[t]$ is free as an $\mathcal{R}$-module.

Given $M \in \mathcal{H}_{1,S_+}(\mathcal{R}[t])$, let $f_M$ denote the endomorphism of $M$ induced by the $\mathcal{R}[t]$-module structure of $M$ (so, $f_M(x) = x \cdot t$). Let $\pi_M \colon \eta(M) \to M$ be the $\mathcal{R}[t]$ module homomorphism such that $\pi_M \colon x \otimes t^i \mapsto (f_M)^i(x)$, for $i$ in $\mathbb{Z}_+$. Recall $j \colon \mathcal{H}_{1,S_+}(\mathcal{R}[t]) \to \mathcal{H}_1(\mathcal{R}[t])$ denotes the inclusion functor. For morphisms $\psi \colon A \to B$ in $\mathcal{H}_{1,S_+}(\mathcal{R}[t])$, we define transformations of functors, $\mathcal{F} \colon \eta \mapsto \eta$ and $\mathcal{G} \colon \eta \mapsto j$, by the following commutative diagrams of $\mathcal{R}[t]$-module homomorphisms,

$$\begin{tikzcd} \eta(A) \arrow[r, "\eta(\psi)"] \arrow[d, "\mathcal{F}(A)"'] & \eta(B) \arrow[d, "\mathcal{F}(B)"] \\ \eta(A) \arrow[r, "\eta(\psi)"] & \eta(B) \end{tikzcd} \quad = \quad \begin{tikzcd} A \otimes_{\mathcal{R}} \mathcal{R}[t] \arrow[r, "\psi\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes t - f_A \otimes \mathrm{id}"'] & B \otimes_{\mathcal{R}} \mathcal{R}[t] \arrow[d, "\mathrm{id}\otimes t - f_B \otimes \mathrm{id}"] \\ A \otimes_{\mathcal{R}} \mathcal{R}[t] \arrow[r, "\psi\otimes\mathrm{id}"] & B \otimes_{\mathcal{R}} \mathcal{R}[t] \end{tikzcd}$$

and

$$\begin{tikzcd} \eta(A) \arrow[r, "\eta(\psi)"] \arrow[d, "\mathcal{G}(A)"'] & \eta(B) \arrow[d, "\mathcal{G}(B)"] \\ j(A) \arrow[r, "j(\psi)"] & j(B) \end{tikzcd} \quad = \quad \begin{tikzcd} A \otimes_{R} \mathcal{R}[t] \arrow[r, "\psi\otimes\mathrm{id}"] \arrow[d, "\pi_A"'] & B \otimes_{\mathcal{R}} \mathcal{R}[t] \arrow[d, "\pi_B"] \\ A \arrow[r, "\psi"] & B \end{tikzcd}$$

Because the vertical arrows do not depend on $\psi$, $\mathcal{F}$ and $\mathcal{G}$ are natural transformations. Also $\eta \overset{\mathcal{F}}{\rightarrowtail} \eta \overset{\mathcal{G}}{\twoheadrightarrow} j$ is a short exact sequence of functors since for any $M \in \mathcal{H}_{1,S_+}(\mathcal{R}[t])$, the sequence

$$0 \longrightarrow M \otimes_{\mathcal{R}} \mathcal{R}[t] \xrightarrow{t - f_M} M \otimes_{\mathcal{R}} \mathcal{R}[t] \xrightarrow{\pi_M} M \longrightarrow 0$$

(with $t - f_M \colon x \otimes t^i \mapsto x \otimes t^{i+1} - f_M(x) \otimes t^i$) is exact (see e.g. [40, p. 630]). Let $K_1(\eta), K_1(j)$ denote the corresponding maps on K-theory. Because $\eta \overset{\mathcal{F}}{\rightarrowtail} \eta \overset{\mathcal{G}}{\twoheadrightarrow} j$ is a short exact sequence of exact functors of exact categories, it follows from the Additivity Theorem [19, V.1.2] that $K_1(\eta) = K_1(\eta) + K_1(j)$. Thus $K_1(j)$ is the zero map. This concludes the proof of Theorem 2.1 in the case $\mathcal{R}$ is commutative.

*Remark* 2.11. In the commutative case, the injectivity of $K_1(\mathcal{R}[t]) \to K_1(S_+^{-1}\mathcal{R}[t])$ may also be deduced using an argument of Grayson, found in [39, Corollary 6]. As described in [39, Corollary 6], one constructs a Mayer-Vietoris sequence that splits up, analogous to the proof of the Fundamental Theorem concerning $K_1(\mathcal{R}[t, t^{-1}])$ as found in [41, p.20].

### The General Case

From here on, we do not assume the ring $\mathcal{R}$ is commutative. Before proving the general case of Theorem 2.1, we present the necessary material from [42, 43].

**Definition 2.12.** Let $\Sigma = \{\sigma_i\}$ be a collection of monomorphisms between finitely generated projective $\mathcal{R}$-modules. The exact category $\mathcal{E} = \mathcal{E}(\Sigma)$ is defined to be the full subcategory of $\mathcal{H}_1(\mathcal{R})$ determined by the following conditions:

1. For every $\sigma \in \Sigma$, $\mathrm{coker}(\sigma)$ lies in $\mathcal{E}$.

2. If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of objects in $\mathcal{H}_1(\mathcal{R})$ such that two of the objects $M_1, M_2, M_3$ lie in $\mathcal{E}$, then so does the third.

3. $\mathcal{E}$ contains all direct summands of its objects.

4. $\mathcal{E}$ is minimal, subject to (1),(2) and (3).

Following [42], we refer to the objects in the category $\mathcal{E}(\Sigma)$ as $(\mathcal{R}, \Sigma)$-torsion modules. When the collection $\Sigma$ is clear, we may simply refer to $\mathcal{E}$ instead of $\mathcal{E}(\Sigma)$. Note that in Definition 2.12 we have used $\mathcal{H}_1(\mathcal{R})$ in place of the category of all finitely presented $\mathcal{R}$-modules of projective dimension $\leq 1$ in [42]. The two definitions are equivalent, because the category $\mathcal{H}_1(\mathcal{R})$ and the category of finitely presented modules of projective dimension $\leq 1$ coincide: given a finitely presented module $M$ of projective dimension less than or equal to one, one may always construct a resolution of length one or less by finitely generated projective modules [19, 4.1.6].

The next theorem will not be used directly, but helps provide context for the torsion category $\mathcal{E}$ defined above, so we include it.

**Theorem 2.2.** *[42, Proposition 0.7] Assume for all $\sigma \in \Sigma$ that $\sigma$ is a monomorphism, and let $\mathcal{E} = \mathcal{E}(\Sigma)$ be as in Definition 2.12. Then an $\mathcal{R}$-module $M$ belongs to $\mathcal{E}$ iff*

*(i) $M$ is finitely presented with projective dimension $\leq 1$, and*

*(ii) $\{\Sigma^{-1}\mathcal{R}\} \otimes_{\mathcal{R}} M$ and $Tor_1^{\mathcal{R}}(\Sigma^{-1}\mathcal{R}, M)$ both vanish.*

When $\mathcal{R}$ is commutative and $S \subset \mathcal{R}$ is a multiplicatively closed set of non-zero-divisors, we let $\Omega_S$ denote the collection of all homomorphisms $f_s \colon \mathcal{R} \to \mathcal{R}$ given by $f_s \colon x \mapsto xs$, with $s \in S$. In this case the Cohn localization $\Omega_S^{-1}\mathcal{R}$ coincides with the standard commutative localization $S^{-1}\mathcal{R}$, and $\mathcal{E}(\Omega_S)$ agrees with $\mathcal{H}_{1,S}(\mathcal{R})$. Indeed, in the commutative case $S^{-1}\mathcal{R}$ is flat, so we always have $Tor_1^{\mathcal{R}}(S^{-1}\mathcal{R}, M) = 0$, and for a nontrivial finitely generated $\mathcal{R}$-module $M$, $S^{-1}\mathcal{R} \otimes_{\mathcal{R}} M = 0$ iff there exists $s \in S$ such that $Ms = 0$.

The following theorem is the main tool we use to prove the injectivity of the map $K_1(\mathcal{R}[t]) \to K_1(\Omega_+^{-1}\mathcal{R}[t])$. The sequence 2.13, without the leftmost map, was established by Schofield in [23]. The extension to include the term $K_1(\mathcal{E}) \to K_1(\mathcal{R})$, which is critical for our application, is due to Neeman and Ranicki; Theorem 2.3 is a combination of [42, Theorem 0.5] and the result stated as Theorem 2.5 below.

**Theorem 2.3.** *[42, p. 789] Let $\mathcal{R}$ be a ring, and $\Sigma$ be a collection of monomorphisms between finitely-generated projective $\mathcal{R}$-modules. Let $\mathcal{E} = \mathcal{E}(\Sigma)$ denote the torsion category of Definition 3.2. Then there is an exact sequence*

$$K_1(\mathcal{E}) \to K_1(\mathcal{R}) \to K_1(\Sigma^{-1}\mathcal{R}) \to K_0(\mathcal{E}) \to K_0(\mathcal{R}) \to K_0(\Sigma^{-1}\mathcal{R}) \tag{2.13}$$

*Remark* 2.14. Neeman and Ranicki [43] extended (2.13) to

$$\cdots \to K_n(\mathcal{E}) \to K_n(\mathcal{R}) \to K_n(\Sigma^{-1}\mathcal{R}) \to K_{n-1}(\mathcal{E}) \to \cdots$$

for all $n > 1$ under the hypothesis that the localization $\Sigma^{-1}\mathcal{R}$ is *stably flat*: for all $n \geq 1$ the group $\mathrm{Tor}_n^{\mathcal{R}}(\Sigma^{-1}\mathcal{R}, \Sigma^{-1}\mathcal{R})$ vanishes. The six term version (2.13) has no stably flat requirement. We have no need of the full long exact in the present paper.

By Theorem 2.3, to prove the injectivity of $K_1(\mathcal{R}[t]) \to K_1(\Omega_+^{-1}\mathcal{R}[t])$ it is sufficient to show the map $K_1(\mathcal{E}) \to K_1(\mathcal{R}[t])$ in (2.13) is zero. For this, we will need a more detailed examination of the original sequence from [43, Corollary 4.9]. Definitions of maps in (2.13) involve identifications of various groups, and we take care to track through these identifications. We do this for general $\Sigma$ at first, specializing to the case of interest ($\Sigma = \Omega_+$, the Fredholms) at a later point.

Recall that a Waldhausen category consists of a category with a subcategory of morphisms called cofibrations, along with a distinguished family of morphisms called weak equivalences, satisfying some axioms, which may be found in [19, Definition II.9.1.1]. We let $C_b(\mathrm{Proj}\mathcal{R})$ denote the following Waldhausen category:

1. The objects are bounded chain complexes of finitely generated projective $\mathcal{R}$-modules
2. The morphisms are chain maps
3. The cofibrations are degree-wise split monomorphisms
4. The weak equivalences are the quasi-isomorphisms, i.e. the chain maps inducing an isomorphism on homology in every degree.

The only Waldhausen categories which will be considered in this article are full subcategories of the category of chain complexes over some exact category, where the morphisms are chain maps, the cofibrations are degree-wise split monomorphisms, and the weak equivalences are quasi-isomorphisms.

For an exact category $\mathcal{A}$ or Waldhausen category $\mathcal{B}$, we let $K(\mathcal{A})$ and $K(\mathcal{B})$ denote the corresponding K-theory spaces, as in [19, IV.6.3 and IV.8.4]. For a topological space $X$, let $\pi_n(X)$ denote the $n$th homotopy group. By definition, $K_n(\mathcal{A}) = \pi_n(K(\mathcal{A}))$, and $K_n(\mathcal{B}) = \pi_n(K(\mathcal{B}))$. Since the definitions agree in the case $\mathcal{B}$ is exact [19, IV.8.6], we do not distinguish, and use the same $K(\mathcal{A})$ and $K(\mathcal{B})$ for both.

We will make use of the following theorem.

**Theorem 2.4** (Gillet-Waldhausen)**.** *Let $\mathcal{A}$ be an exact category, closed under taking kernels of surjections. Then the exact monomorphism $\mathcal{A} \hookrightarrow C_b(\mathcal{A})$, taking an object $M$ to the chain complex which is $M$ in degree 0 and is zero elsewhere, induces a homotopy equivalence $K(\mathcal{A}) \xrightarrow{\sim} K(C_b(\mathcal{A}))$, and hence isomorphisms $K_n(\mathcal{A}) \xrightarrow{\cong}$*

$K_n(C_b(\mathcal{A}))$.

A proof of Theorem 2.4 may be found in [19, V.2.2, II.9.2.2].

Let $\Sigma = \{\sigma_i\}$ denote a collection of morphisms between finitely generated projective $\mathcal{R}$-modules. Note that each $\sigma \in \Sigma$ may be considered in $C_b(\text{Proj}\mathcal{R})$ as the complex

$$\cdots \to 0 \to P \xrightarrow{\sigma} Q \to 0 \cdots \tag{2.15}$$

with $P, Q$ in degrees $0, 1$ and modules in all other degrees zero.

By a Waldhausen subcategory $\mathcal{A} \subset \mathcal{B}$ of a Waldhausen category $\mathcal{B}$ we mean a subcategory $\mathcal{A} \subset \mathcal{B}$ which is also a Waldhausen category, satisfying:

1. the inclusion functor $\mathcal{A} \to \mathcal{B}$ is exact, i.e. preserves all of the following: zero, cofibrations, weak equivalences, and pushouts along cofibrations,

2. the cofibrations in $\mathcal{A}$ are the maps in $\mathcal{A}$ which are cofibrations in $\mathcal{B}$ and whose cokernels lie in $\mathcal{A}$,

3. the weak equivalences in $\mathcal{A}$ are the weak equivalences of $\mathcal{B}$ which lie in $\mathcal{A}$.

Define a Waldhausen category as follows:

**Definition 2.16.** The category $\mathbf{R}$ is the smallest subcategory of $C_b(\text{Proj}\mathcal{R})$ which:

(i) contains the complex (2.15) as defined above, for all $\sigma \in \Sigma$,

(ii) contains all acyclic complexes,

(iii) is closed under the formation of mapping cones and suspensions,

(iv) contains any direct summand of any of its objects.

The following theorem is a combination of [43, Corollary 4.9] and [42, Theorem 0.10].

**Theorem 2.5.** *[42, p.789] Let $\mathcal{R}$ be a ring, and $\Sigma$ a collection of homomorphisms between finitely generated projective $\mathcal{R}$-modules. There is an exact sequence*

$$K_1(\boldsymbol{R}) \to K_1(C_b(\mathrm{Proj}\mathcal{R})) \to K_1(\Sigma^{-1}\mathcal{R}) \to K_0(\boldsymbol{R}) \to K_0(C_b(\mathrm{Proj}\mathcal{R})) \to K_0(\Sigma^{-1}\mathcal{R}) \tag{2.17}$$

In Theorem 2.5, $\mathcal{R}$ is general and there is no requirement that $\Sigma$ consists of monomorphisms. The maps $K_i(\mathbf{R}) \to K_i(C_b(\mathrm{Proj}\mathcal{R}))$ are induced by the inclusion $\mathbf{R} \to C_b(\mathrm{Proj}\mathcal{R})$. Upon replacing $C_b(\mathrm{Proj}\mathcal{R})$ in Theorem 2.5 with $\mathcal{R}$ using Gillet-Waldhausen, the maps $K_i(\mathcal{R}) \to K_i(\Sigma^{-1}\mathcal{R})$ coincide with the maps $K_i(\mathcal{R}) \to K_i(\Sigma^{-1}\mathcal{R})$ induced by the ring homomorphism $\mathcal{R} \to \Sigma^{-1}\mathcal{R}$ (see the discussion following Theorem 0.10 in [42]).

Let $C_b(\mathcal{H}_1(\mathcal{R}))$ denote the Waldhausen category of bounded chain complexes of finitely presented $\mathcal{R}$-modules of projective dimension $\leq 1$. Given $\Sigma$ a collection of monomorphisms and $\mathcal{E} = \mathcal{E}(\Sigma)$ as in Definition 2.12, we let $C_b(\mathcal{E})$ denote the Waldhausen category of bounded chain complexes of objects of $\mathcal{E}$. For both $C_b(\mathcal{H}_1(\mathcal{R}))$ and $C_b(\mathcal{E})$, the cofibrations consist of the chain maps which are degree-wise split monomorphisms, and the weak equivalences are the quasi-isomorphisms.

**Lemma 2.18.** [42, Theorem 2.7] There is a Waldhausen subcategory $\mathbf{R}' \subset C_b(\mathcal{H}_1(\mathcal{R}))$ and inclusions $\mathbf{R} \to \mathbf{R}'$, $C_b(\mathcal{E}) \to \mathbf{R}'$ that induce homotopy equivalences

$$\begin{tikzcd}
K(\mathbf{R}) \arrow[dr, "\simeq"] & \\
& K(\mathbf{R}') \\
K(C_b(\mathcal{E})) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

*Remark* 2.19. The subcategory $\mathbf{R}'$ of Lemma 2.18 defined in [42, Theorem 2.7] is the full Waldhausen subcategory of $C_b(\mathcal{H}_1(\mathcal{R}))$ consisting of all objects which become isomorphic in $D(C_b(\mathcal{H}_1(\mathcal{R})))$ to objects in the image of $D(\mathbf{R})$, the derived category of $\mathcal{R}$. Details regarding $\mathbf{R}'$ are not important for the present article, and may be found in the proof of Theorem 2.7 in [42].

One consequence of 2.18 is that, by the Gillet-Waldhausen theorem, we have $K(\mathbf{R}) \simeq K(\mathcal{E})$, which gives one of the identifications made when passing between 2.3 and 2.5.

We now specialize to the case of interest, in order to prove the main result of the section. *For the remainder of the section, we let $\Sigma = \Omega_+$ denote the collection of Fredholm homomorphisms of finitely generated projective $\mathcal{R}[t]$-modules.*

*Proposition* 2.20. Consider a polynomial ring $\mathcal{R}[t]$, with $\Omega_+$ the collection of Fredholm homomorphisms, and $\mathbf{R}$ as defined in Definition 3.8. Then the maps

$$K_n(i)\colon K_n(\mathbf{R}) \to K_n(C_b(\mathrm{Proj}\mathcal{R}[t]))$$

are zero, for all $n$, where $K_n(i)$ is the map induced by the inclusion $\mathbf{R} \to C_b(\mathrm{Proj}\mathcal{R}[t])$.

Since the maps $K_n(\mathbf{R}) \to K_n(C_b(\mathrm{Proj}\mathcal{R}[t]))$ in Theorem 2.5 are induced by the inclusion $\mathbf{R} \to C_b(\mathrm{Proj}\mathcal{R}[t])$, Theorem 2.1 will follow from Proposition 2.20.

*Proof of Proposition 2.20:* Consider the diagram of inclusions

$$\begin{tikzcd} & C_b(\mathcal{H}_1(\mathcal{R}[t])) \\ \mathbf{R} \arrow[ur] \arrow[r] & C_b(\mathrm{Proj}\mathcal{R}[t]) \arrow[u] \end{tikzcd}$$

By the Resolution Theorem (see [19, V.3.1]) we have $K(\mathrm{Proj}\mathcal{R}[t]) \simeq K(\mathcal{H}_1(\mathcal{R}[t]))$, so combined with Gillet-Waldhausen, the vertical functor on the right induces a homotopy equivalence

$$K(C_b(\mathrm{Proj}\mathcal{R}[t])) \xrightarrow{\simeq} K(C_b(\mathcal{H}_1(\mathcal{R}[t])))$$

and therefore isomorphisms $K_n(C_b(\mathrm{Proj}\mathcal{R}[t])) \to K_n(C_b(\mathcal{H}_1(\mathcal{R}[t])))$ for all $n$. Furthermore, Lemma (2.18) shows that the images of the homomorphisms

$$K_n(\mathbf{R}) \to K_n(C_b(\mathcal{H}_1(\mathcal{R}[t])))$$

$$K_n(C_b(\mathcal{E})) \to K_n(C_b(\mathcal{H}_1(\mathcal{R}[t])))$$

coincide. We claim that the map $K_n(C_b(\mathcal{E})) \to K_n(C_b(\mathcal{H}_1(\mathcal{R}[t])))$ is zero for all $n$. This will prove that $K_1(\mathcal{R}[t]) \to K_1(\Omega_+^{-1}\mathcal{R}[t])$ is injective, in light of Theorem 3.8. We have a diagram

$$\begin{tikzcd} C_b(\mathcal{E}) \arrow[r] & C_b(\mathcal{H}_1(\mathcal{R}[t])) \\ \mathcal{E} \arrow[u] \arrow[r] & \mathcal{H}_1(\mathcal{R}[t]) \arrow[u] \end{tikzcd}$$

in which by Gillet-Waldhausen the vertical arrows induce homotopy equivalences in $K$,

$$\begin{tikzcd} K(C_b(\mathcal{E})) \arrow[r] & K(C_b(\mathcal{H}_1(\mathcal{R}[t]))) \\ K(\mathcal{E}) \arrow[u, "\simeq"] \arrow[r] & K(\mathcal{H}_1(\mathcal{R}[t])) \arrow[u, "\simeq"'] \end{tikzcd}$$

Thus it suffices to show that the maps $K_n(\mathcal{E}) \to K_n(\mathcal{H}_1(\mathcal{R}[t]))$, induced by the inclusion functor $j\colon \mathcal{E} \to \mathcal{H}_1(\mathcal{R}[t]))$, are zero for all $n$.

Let $X$ be the full subcategory of $\mathcal{H}_1(\mathcal{R}[t])$ whose objects are the modules $M$ in (i.e. the objects $M$ of) $\mathcal{H}_1(\mathcal{R}[t])$ such that $\eta(M) := M \otimes_{\mathcal{R}} \mathcal{R}[t]$ is in $\mathcal{H}_1(\mathcal{R}[t])$. (For example, $\mathcal{R}[t]$ is in $\mathcal{H}_1(\mathcal{R}[t])$ but is not in $X$, because $\mathcal{R}[t]$ is not finitely generated as an $\mathcal{R}$-module.) We claim that $\mathcal{E}$ is contained in $X$. Consider each of the following:

1. If $\sigma \in \Omega_+$, then $\mathrm{coker}(\sigma)$ is finitely generated projective as an $\mathcal{R}$-module, since $\Omega_+$ consists of Fredholm morphisms. It follows that $\mathrm{coker}(\sigma) \otimes_{\mathcal{R}} \mathcal{R}[t]$ lies in $\mathrm{Proj}\mathcal{R}[t] \subset \mathcal{H}_1(\mathcal{R}[t])$, so $X$ contains the cokernels of all morphisms $\sigma \in \Omega_+$.

2. Now suppose

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact in $\mathcal{H}_1(\mathcal{R}[t])$.

Tensoring this sequence with $\mathcal{R}[t]$ gives

$$0 \to M_1 \otimes_{\mathcal{R}} \mathcal{R}[t] \to M_2 \otimes_{\mathcal{R}} \mathcal{R}[t] \to M_3 \otimes_{\mathcal{R}} \mathcal{R}[t] \to 0 \tag{2.21}$$

which is an exact sequence of $\mathcal{R}[t]$-modules, since $\mathcal{R}[t]$ is free over $\mathcal{R}$. We claim that if two of $M_1, M_2, M_3$ lie in $X$, then so does the third.

(a) Suppose $M_2$ and $M_3$ lie in $X$. Then $\eta(M_2)$ and $\eta(M_3)$ lie in $\mathcal{H}_1(\mathcal{R}[t])$. Since $\mathcal{H}_1(\mathcal{R}[t])$ is closed under kernels of surjections (see [19, II.7.7.1]), the exactness of (2.21) shows that $M_1$ lies in $X$.

(b) Suppose $M_1$ and $M_3$ lie in $X$. Then the exactness of (2.21) along with the fact that $\mathcal{H}_1(\mathcal{R}[t])$ is closed under extensions (see [44, 2.2.8]) implies $M_2$ lies in $X$ as well.

(c) Suppose $M_1$ and $M_2$ lie in $X$. Then the exactness of (2.21) shows that $\eta(M_3)$ is finitely presented, being a quotient of two finitely presented modules. But it is clear that $\eta(M_3)$ is also of homological dimension $\leq 1$, so $M_3$ is in $X$ as well.

3. $X$ contains all direct summands of its objects, since $\mathcal{H}_1(\mathcal{R}[t])$ is closed under direct summands.

Since $\mathcal{E}$ is the minimal subcategory of $\mathcal{H}_1(\mathcal{R}[t])$ satisfying the corresponding properties (1,2,3) in Definition 2.12, we have $\mathcal{E} \subset X$, as desired.

The remainder of the proof closely follows that of the commutative case given earlier. Given $M \in \mathcal{E}$, let $f_M$ denote the endomorphism of $M$ induced by the $\mathcal{R}[t]$-module structure of $M$ (so $f_M(x) = t \cdot x$). From the discussion above we have the exact functor $\eta\colon \mathcal{E}(\Omega_+) \to \mathcal{H}_1(\mathcal{R}[t])$, and we denote by $\mathcal{F}$ the natural transformation $\mathcal{F}\colon \eta \mapsto \eta$ defined by $\mathcal{F}(M)\colon \eta(M) \overset{t-f_M}{\to} \eta(M)$. Recall $j\colon \mathcal{E} \to \mathcal{H}_1(\mathcal{R}[t])$ denotes the inclusion functor. Define the natural transformation $\mathcal{G}\colon \eta \mapsto j$ by $\mathcal{G}\colon \eta(M) \overset{\pi}{\to} M$, where $\pi(p(t) \otimes x) = p(f_M)(x)$. Then $\eta \overset{\mathcal{F}}{\rightarrowtail} \eta \overset{\mathcal{G}}{\twoheadrightarrow} j$ is an exact sequence of functors, since for any $M \in \mathcal{E}$, the sequence

$$0 \to M \otimes_{\mathcal{R}} \mathcal{R}[t] \overset{t-f_m}{\rightarrowtail} M \otimes_{\mathcal{R}} \mathcal{R}[t] \overset{\pi}{\twoheadrightarrow} M \to 0$$

is exact (see [40, p. 630]). Letting $K_n(\eta), K_n(j)$ denote the corresponding maps on K-theory, the Additivity Theorem (V.1.2 in [19]) now implies that, for all $n$, $K_n(\eta) = K_n(\eta) + K_n(j)$. Thus $K_n(j)$ is the zero map, for all $n$. This finishes the proof of 2.1.

## 2.4 The elementary stabilizer

Recall our notational conventions (2.5, 2.6). In particular, $\mathfrak{M}(\mathcal{R})$ is the set of $\mathbb{N} \times \mathbb{N}$ matrices over the ring $\mathcal{R}$ equal to the identity except in finitely many entries, with $\mathrm{El}(\mathcal{R}) \subset \mathrm{GL}(\mathcal{R}) \subset \mathfrak{M}(\mathcal{R})$. Given $\mathcal{R}$ and $M$ in $\mathfrak{M}(\mathcal{R})$, the elementary stabilizer of $M$ is defined to be

$$\mathrm{ElSt}_{\mathcal{R}}(M) = \{U \in \mathrm{GL}(\mathcal{R}) : U\mathrm{Orb}_{\mathrm{El}(\mathcal{R})}(M) \subset \mathrm{Orb}_{\mathrm{El}(\mathcal{R})}(M)\} \ . \tag{2.22}$$

Because $\mathrm{El}(\mathcal{R})$ is a subgroup of $\mathrm{ElSt}_{\mathcal{R}}(M)$, $\{[U] \in K_1(\mathcal{R})\colon U \in \mathrm{ElSt}_{\mathcal{R}}(M)\}$ is a subgroup of $K_1(\mathcal{R})$, which by abuse of notation we also denote by $\mathrm{ElSt}_{\mathcal{R}}(M)$. We give a shorter notation for the elementary stabilizer which is our main interest. Given an $n \times n$ matrix $A$ over $\mathcal{R}$, let $I - tA$ denote $(I_n - tA)_{\mathrm{st1}} = I - tA_{\mathrm{st0}}$ (as in 2.6) and define

$$E(A, \mathcal{R}) := \mathrm{ElSt}_{\mathcal{R}[t]}(I - tA) \ . \tag{2.23}$$

If $U \in E(A, \mathcal{R})$, then there are $E, F$ from $\mathrm{El}(\mathcal{R}[t])$ such that $U(I-tA) = E(I-tA)F$. Evaluating at $t = 0$, we see that $E(A, \mathcal{R})$, considered as a subgroup of $K_1(\mathcal{R}[t])$, satisfies

$$E(A, \mathcal{R}) \subset NK_1(\mathcal{R}) \ . \tag{2.24}$$

*Remark* 2.25. Suppose $\mathcal{R}$ is a commutative ring for which $\mathrm{NK}_1(\mathcal{R})$ is nontrivial but the embedding $\mathrm{SK}_1(\mathcal{R}) \to \mathrm{SK}_1(\mathcal{R}[t])$ induced by the inclusion $\mathcal{R} \to \mathcal{R}[t]$ is surjective. (For example, $\mathcal{R} = \mathcal{S}[x]/(x^N)$, with $N > 1$ and $\mathcal{S}$ a commutative regular ring [19, Example III.3.8.1].) Then the containment of (2.24) is proper, and $E(A, \mathcal{R})$ is trivial for every matrix $A$ over $\mathcal{R}$.

*Proposition* 2.26. Suppose $\mathcal{R}$ is a ring and $A \in \mathfrak{M}(\mathcal{R})$. Then there is a bijection

$$\begin{aligned} K_1(\mathcal{R})/\mathrm{ElSt}_{\mathcal{R}}(A) &\to \{\mathrm{Orb}_{\mathrm{El}(\mathcal{R})}(B) : B \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)\} \\ [U] &\mapsto U\mathrm{Orb}_{\mathrm{El}(\mathcal{R})}(A) \ . \end{aligned}$$

If $B \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)$, then $\mathrm{ElSt}_{\mathcal{R}}(B) = \mathrm{ElSt}_{\mathcal{R}}(A)$.

*Proof.* For $B \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)$, let $\mathcal{O}_B = \mathrm{Orb}_{\mathrm{El}(\mathcal{R})}(B)$. Then $U\mathcal{O}_B = \mathcal{O}_{UB} = \mathcal{O}_{BU} = \mathcal{O}_B U$, for all $U$ in $\mathrm{GL}(\mathcal{R})$ and $B \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)$. Therefore the rule $U \colon \mathcal{O} \mapsto U\mathcal{O}$ gives a well defined action of $\mathrm{GL}(\mathcal{R})$ on $\{\mathcal{O}_B \colon B \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)\}$. The isotropy group of an element $\mathcal{O}_B$ under this action is $\mathrm{ElSt}_{\mathcal{R}}(B)$, which contains $\mathrm{El}(\mathcal{R})$. Therefore given $B \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)$ we have well defined bijections

$$\begin{array}{ccccc} K_1(\mathcal{R})/\mathrm{ElSt}_{\mathcal{R}}(B) & \to & \mathrm{GL}(\mathcal{R})/\mathrm{ElSt}_{\mathcal{R}}(B) & \to & \{\mathcal{O}_C \colon C \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)\} \\ [U] & \mapsto & [U] & \mapsto & U\mathcal{O}_C \end{array} .$$

For $B \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)$, the isotropy groups $\mathrm{ElSt}_{\mathcal{R}}(A)$ and $\mathrm{ElSt}_{\mathcal{R}}(B)$ are conjugate in $\mathrm{GL}(\mathcal{R})$, and therefore equal, as $\mathrm{ElSt}_{\mathcal{R}}(A)$ contains $\mathrm{El}(\mathcal{R})$, the commutator subgroup of $\mathrm{GL}(\mathcal{R})$. $\square$

The next result, a key fact for us, follows directly from Theorem 2.1. For its statement, recall that by our notational convention, the elementary stabilizer of a finite matrix $I - A$ means the elementary stabilizer of $(I - A)_{\mathrm{st1}}$. Recall that the map $i : \mathcal{R}[t] \to \Omega_+^{-1}\mathcal{R}[t]$ denotes the standard map coming from the definition of the localization.

**Theorem 2.6.** *Let $\mathcal{R}[t]$ be a polynomial ring, with coefficient ring $\mathcal{R}$. If $A$ is a square matrix over $\mathcal{R}[t]$ such that $I - A \in \Omega_+(\mathcal{R}[t])$, then* $\mathrm{ElSt}_{\mathcal{R}[t]}(I - A)$ is trivial in $K_1(\mathcal{R}[t])$.

*Proof.* If $I - A \in \Omega_+(\mathcal{R}[t])$ and $U \in \mathrm{ElSt}_{\mathcal{R}[t]}(I - A)$, then $[U]$ is in the kernel of the map $i_* : K_1(\mathcal{R}[t]) \to K_1(\Omega_+^{-1}\mathcal{R}[t])$ of Theorem 2.1. By Theorem 2.1, this implies $[U] = 0$. $\square$

**Theorem 2.7.** *Let $A$ be a square matrix over $t\mathcal{R}[t]$ such that $A = \sum_{i=1}^{d} t^i A_i$. Suppose $A$ satisfies any of the following:*

1. *$A_d$ is nilpotent and $A_i = 0$ for $1 \leq i < d$.*

2. *$A_d$ is invertible over $\mathcal{R}$.*

3. *$A_d$ is idempotent and $A_i = 0$ for $1 \leq i < d$.*

*Then* $\mathrm{ElSt}_{\mathcal{R}[t]}(I - A)$ *is trivial in $K_1(\mathcal{R}[t])$.*

*Proof.* The claim for case (1) is clear, because $I - A \in \mathrm{GL}(\mathcal{R}[t])$. For the remaining cases, by Theorem 2.6 it suffices to show that the matrix $I - A$ is invertible over $\Omega_+^{-1}(\mathcal{R}[t])$. For case (2), the matrix $(I - A)A_d^{-1}$ is monic, and hence invertible over $\Omega_+^{-1}\mathcal{R}[t]$.

For case (3), we first note that if $P$ is an $n \times n$ idempotent matrix over $\mathcal{R}$, then $cok(I - tP)$ is a finitely generated projective $\mathcal{R}$-module. Let $J$ denote $P(\mathcal{R}^n)$, the image of the $\mathcal{R}$-module endomorphism $\mathcal{R}^n \xrightarrow{P} \mathcal{R}^n$ given by multiplication by $P$. The finitely generated $\mathcal{R}$-module $J$ is projective, since $P$ is idempotent. Letting $x_0, \ldots, x_{d-1}$ denote elements of $\mathcal{R}^n$, we have an isomorphism of $\mathcal{R}$-modules $cok(I - tP) \to J^d$ given by $[\sum_{i=0}^{d-1} t^{i+d} P x_i] \mapsto (Px_0, \ldots, Px_{d-1})$. It follows that the matrix $I - t^d P$ belongs to $\Omega_+^{mat}(\mathcal{R}[t])$, i.e. is Fredholm. Thus, for the map $i : \mathcal{R}[t] \to \Omega_+^{-1}\mathcal{R}[t]$ given by the localization, the matrix $i(I - t^d P)$ is invertible over $\Omega_+^{-1}\mathcal{R}[t]$.

□

Given $A$ square over $\mathcal{R}$, we would like to know the structure of $E(A, \mathcal{R})$. We cannot answer the first question about this:

*Question* 2.27. Must $E(A, \mathcal{R})$ be trivial, for every square matrix $A$ over $\mathcal{R}$?

*Remark* 2.28. Version 1 of our arXiv post [45] claimed an affirmative answer to Question 2.27, but the proof contained an error: Corollary 3.20 is wrong. The error in its proof is the claim of existence of the map $f_1$. Under $j$, a monic matrix is carried to a reverse monic matrix, which need not be invertible in $\Omega_+^{-1}\mathcal{R}[t]$; so we cannot apply the universal property of the Cohn localization to produce $f_1$.

In the case $\mathcal{R}$ is commutative, localization techniques allow us to make further statements regarding $\mathrm{ElSt}_{\mathcal{R}[t]}(I-A)$ for certain matrices $A$. Our main tool for this will be the following result of Vorst (see [46, 1.7, Remark 1.12]). For an element $r \in \mathcal{R}$ we let $\mathcal{R}_r$ denote the localization of $\mathcal{R}$ at the multiplicative subset $\{r^i\}$. Recall that a collection of elements $\{f_1, \ldots, f_k\} \subset \mathcal{R}$ is called a unimodular row if the ideal $(f_1, \ldots, f_k)$ generated by the collection is $\mathcal{R}$ itself.

**Theorem 2.8.** *[46, Corollary 1.7] Let $\mathcal{R}$ be a commutative ring, and let $f_1, \ldots, f_k \in \mathcal{R}$ be a unimodular row over $\mathcal{R}$. Then the map $NK_1(\mathcal{R}) \to \prod_{i=1}^{r} NK_1(\mathcal{R}_{f_i})$ is injective.*

*Proposition* 2.29. Let $\mathcal{R}$ be a commutative ring, and let $A$ be a square matrix over $\mathcal{R}$ such that $0 \neq \det(A)$ and $\det(A)$ is not a zero-divisor. Suppose there exists $j$ unimodular rows

$$\{\det(A), f_{1,1}, \ldots, f_{k_1,1}\}, \ldots, \{\det(A), f_{1,j}, \ldots, f_{k_j,j}\}$$

each containing $\det(A)$ such that

$$\bigcup_{i=1}^{j} \bigcap_{n=1}^{k_i} \ker(NK_1(\mathcal{R}) \to NK_1(\mathcal{R}_{f_{n,i}})) = NK_1(\mathcal{R})$$

Then $\mathrm{ElSt}_{\mathcal{R}[t]}(I-A) = 0$.

*Proof.* Suppose $G \in \mathrm{ElSt}_{\mathcal{R}[t]}(I-A)$, and let $[G]$ denote its class in $NK_1(\mathcal{R})$. The assumptions give an $i$ such that $[G] \in \bigcap_{n=1}^{k_i} \ker(NK_1(\mathcal{R}) \to NK_1(\mathcal{R}_{f_{n,i}}))$. Since $A$ is invertible over $\mathcal{R}_{\det(A)}$, Theorem 2.7 implies $[G] \in \ker(NK_1(\mathcal{R}) \to NK_1(\mathcal{R}_{\det(A)}))$ as well, and hence by Theorem 2.8 we must have $[G] = 0$. $\square$

Proposition 2.29 can occasionally be used to show that for certain matrices $A$, we must have $\mathrm{ElSt}_{\mathcal{R}[t]}(I - A) = 0$. A particular case is demonstrated in the following Corollary.

**Corollary 2.9.** *Let $G$ be a finite abelian group of order $|G|$, and let $\mathbb{Z}G$ denote the integral group ring. Let $A$ be a square matrix over $\mathbb{Z}G$ such that $0 \neq \det(A) = a \in \mathbb{Z}$ and $(a, |G|) = 1$ (so $a$ and the order of $G$ are relatively prime). Then $\mathrm{ElSt}_{\mathcal{R}[t]}(I - A) = 0$.*

*Proof.* The collection $\{a, |G|\}$ forms a unimodular row over $\mathbb{Z}G$. However, by [47, 6.5, pg. 490], $\ker(NK_1(\mathbb{Z}G) \to NK_1((\mathbb{Z}G)_a)) = NK_1(\mathbb{Z}G)$, so Proposition 2.29 implies the claim. $\square$

*Remark* 2.30. The technique of using localization to prove $\mathrm{ElSt}_{\mathcal{R}[t]}(I - A)$ is trivial, as in the proof of Corollary 2.9, has its limits. For example, if $G$ is a finite group, then the map $NK_1(\mathbb{Z}G) \to NK_1((\mathbb{Z}G)_{|G|})$ is the zero map.

*Remark* 2.31. For $G$ a finite group and $A$ a matrix over $\mathbb{Z}G$, the group $K_1(\mathbb{Z}G)/\mathrm{ElSt}_{\mathbb{Z}G}(I - A)$ appeared in [32] as the primary invariant for the classification up to equivariant flow equivalence of certain symbolic dynamical systems: irreducible shifts of finite type with a free continuous shift-commuting $G$-action.

### Limits to generalizations

Theorem 2.7 applies to a rather special class of matrices and its proof appeals to the sophisticated algebraic K-theory of Neeman and Ranicki [42,43]. It is natural to ask if there is an easier proof. It is also natural to hope the conclusion of Theorem

2.7 might hold for a more general class of matrices. We'll note next that some candidate improvements cannot work.

*Remark* 2.32. With an eye to an easier proof, one might note that $(I - A)$ also inverts over the familiar ring of formal power series $\mathcal{R}[[t]]$, and ask if $\mathcal{R}[[t]]$ could play the role of $\Omega_+^{-1}\mathcal{R}[t]$ in Theorem 2.1. We thank Wolfgang Steimle for showing us this fails: the natural map $i^*\colon K_1(\mathcal{R}[t]) \to K_1(\mathcal{R}[[t]])$ induced by the inclusion $i\colon \mathcal{R}[t] \to \mathcal{R}[[t]]$ need not be injective. For example, if $\mathcal{R}$ is commutative, then there is a straightforward decomposition of $K_1(\mathcal{R}[[t]])$ given by

$$0 \to K_1(\mathcal{R}) \to K_1(\mathcal{R}[[t]]) \xrightarrow{d} \hat{W}(\mathcal{R}) \to 0$$

where $\hat{W}(\mathcal{R}) = \{1 + \sum_{i=1}^{\infty} a_i t^i\} \in \mathcal{R}[[t]]$ is the group of Witt vectors. The map $d$ is given by $d(M) = \det(M_0^{-1}M)$, where $M = \sum_{i=0}^{\infty} M_i t^i$ (as in e.g. [25, 14.6]). Thus, if $\mathcal{R}$ is a commutative ring (for example, an integral domain) such that $\det(I - tN) = 1$ for all nilpotent matrices $N$, then the kernel of the map $K_1(\mathcal{R}[t]) \to K_1(\mathcal{R}[[t]])$ induced by the inclusion $\mathcal{R}[t] \to \mathcal{R}[[t]]$ will always contain $NK_1(\mathcal{R})$. Indeed, $d(I - tN) = \det(I - tN) = 1$, so $NK_1(\mathcal{R})$ maps into the kernel of $d$, which is generated by the image of $K_1(\mathcal{R})$; but the only class of the form $[I - tN]$ which lies in the image of $K_1(\mathcal{R})$ is the class $[1]$. Since there are integral domains $\mathcal{R}$ with $NK_1(\mathcal{R}) \neq 0$ (e.g. [35, Example 3.5]) the map $NK_1(\mathcal{R}[t]) \to K_1(\mathcal{R}[[t]])$ need not be injective.

Similarly, one could hope to prove in place of Theorem 2.1 that the map $i_*\colon K_1(\mathcal{R}[t]) \to K_1(S_{RM}^{-1}\mathcal{R}[t])$ is injective, where $S_{RM}$ is the set of reverse monic matrices (those of the form $A = I + \sum_{i=1}^{n} A_i t^i$). But this map need not be injective. In the case $\mathcal{R}$ is commutative, localizing at $S_{RM}$ is equivalent to localizing at $S_{RMP} =$

$\{p(t) = 1 + \sum_{i=1}^{n} a_i t^i\}$, the set of reverse monic polynomials. There is an exact sequence

$$0 \to K_1(\mathcal{R}) \to K_1(S_{RMP}^{-1}\mathcal{R}[t]) \xrightarrow{d} 1 + tS_{RMP}^{-1}\mathcal{R}[t] \to 0$$

which can be found by examining [39, Corollary 3], or [19, III.2.4(2)]. As in the previous paragraph, the map $i_*\colon NK_1(\mathcal{R}) \to K_1(S_{RMP}^{-1}\mathcal{R}[t])$ will fail to be injective for an integral domain $\mathcal{R}$ with $NK_1(\mathcal{R})$ nontrivial.

With regard to generalizing the result, Corollary 2.10 of Proposition 2.33 below shows Theorem 2.7 already fails badly for the more general class of matrices $I - A$ which are injective (in the statement of Corollary 2.10, $\mathcal{R}$ could be a polynomial ring). The rest of this section is devoted to establishing that corollary. We thank David Handelman for showing us the embedding argument which produces the nonderogatory matrix $V = UE$ in the reduction step of Prop. 2.33 below.

*Proposition* 2.33. Suppose $\mathcal{R}$ is an integral domain of characteristic zero which does not embed into $\mathbb{Z}[i]$ or $\mathbb{Z}[e^{i2\pi/3}]$, and $U$ is in $\mathrm{SL}(n, \mathcal{R})$. Then there is an $n \times n$ matrix $A$ over $\mathcal{R}$ such that $I - A$ is injective and $U$ is in the elementary stabilizer $\mathrm{ElSt}_{\mathcal{R}}(I - A)$.

*Proof.* Case I: For this case, we assume there is a matrix $B$ over the field of fractions $\mathbb{F}$ of $\mathcal{R}$ such that $B^{-1}UB = C$, with $C$ a companion matrix. Without loss of generality, we then assume $B$ has all entries in $\mathcal{R}$. Because $C$ must be the companion matrix of the characteristic polynomial of $U$, the entries of $C$ must lie in $\mathcal{R}$. From the companion matrix form and $\det C = 1$, we have $C \in \mathrm{El}(n, \mathcal{R})$. Now $UB = BC$;

defining $A = I - B$, we have that $U$ is in $\mathrm{ElSt}(I - A)$. Clearly $I - A$ is injective.

For the reduction to Case I, it suffices to show that there is a matrix $E \in \mathrm{El}(n, \mathcal{R})$ such that the matrix $V = UE$ has no repeated eigenvalue (and therefore is similar over $\mathbb{F}$ to its companion matrix). After passing if needed to a subring containing the entries of $U$ and still satisfying the nonembeddability hypothesis, we may assume $\mathcal{R}$ is finitely generated. Then $\mathbb{F}$ is isomorphic to an algebraic extension of a subfield of $\mathbb{R}$ (generated by $\mathbb{Q}$ and a set of algebraically independent elements). Thus after embedding $\mathbb{F}$ into $\mathbb{R}$ or $\mathbb{C}$, we have the closure $\overline{\mathbb{F}}$ equal to $\mathbb{R}$ or $\mathbb{C}$. In either case, except under the very special conditions which are excluded in the hypotheses (and are not of interest to us now), the ring $\mathcal{R}$ will likewise be dense in $\overline{\mathbb{F}}$, and consequently $\mathrm{El}(n, \mathcal{R})$ will be dense in $\mathrm{El}(n, \overline{\mathbb{F}}) = \mathrm{Sl}(n, \overline{\mathbb{F}})$. Let $W$ be a matrix in $\mathrm{SL}(n, \mathbb{Z})$ without repeated eigenvalues. The matrices over $\mathbb{F}$ without repeated eigenvalues form a dense open set. Consequently the matrix $U^{-1}W$ in $\mathrm{SL}(n, \mathbb{F})$ can be perturbed to a matrix $E$ in $\mathrm{El}(n, \mathcal{R})$ such that $UE$ has no repeated eigenvalues. $\square$

In the next statement, $E_{A,\mathcal{R}[t]}$ denotes $\{[U] \in K_1(\mathcal{R}[t]) : U \in \mathrm{ElSt}(I - A)\}$.

**Corollary 2.10.** *Suppose $\mathcal{R}$ is a characteristic zero integral domain which is not generated by three elements as an additive group, and $NK_1(\mathcal{R})$ nontrivial. (Such domains exist.) In the class of injective matrices $(I - A)$ over $\mathcal{R}[t]$, the elementary stabilizer $E_{A,\mathcal{R}[t]}$ is not independent of $A$. If $H$ is a finitely generated subgroup of $NK_1(\mathcal{R})$, then there exists an injective $(I - A)$ such that $E_{A,\mathcal{R}[t]}$ contains $H$.*

*Proof.* For $I - A$ invertible over $\mathcal{R}[t]$, $E_{A,\mathcal{R}[t]}$ is trivial in $NK_1(\mathcal{R})$. Now choose $U_k$

in $\mathrm{GL}(\mathcal{R}[t])$ for $1 \leq k \leq K$ with $[U_k] \in NK_1(\mathcal{R})$. Because $\mathcal{R}$ is an integral domain, the $U_k$ lie in $SL(\mathcal{R}[t])$. Proposition 2.33 then gives finite matrices $I - A_k$ over $\mathcal{R}[t]$ with $I - A_k$ injective such that for $A = A_k$, $E_{A,\mathcal{R}[t]}$ contains $U_k$. If $A = \oplus_{k=1}^{K} A_k$, then $E_{A,\mathcal{R}[t]}$ contains all of the $U_k$. For an explicit example of an integral domain $\mathcal{R}$ which embeds into $\mathbb{R}$ and has $NK_1(\mathcal{R}) \neq 0$ see [35, Example 3.5].

$\square$

## 2.5 $\mathrm{SSE}/\mathrm{SE}(A,\mathcal{R}) = \mathrm{NK}_1(\mathcal{R})/E(A,\mathcal{R})$

In this section we prove one of our main results, Theorem 2.13, assuming the main result of the next section, Theorem 4.6.

To begin we state a part of a result from a 1936 paper of Fitting [48]; for an exposition and generalization, we recommend Warfield's paper [49]. A slightly different formulation of Theorem 2.11 is given in [32, Lemma 9.1], with further commentary. We say a $k \times k$ matrix $A$ over $\mathcal{R}$ is injective if matrix multiplication $x \mapsto Ax$ defines an injective map $\mathcal{R}^k \to \mathcal{R}^k$.

**Theorem 2.11.** *[48] Suppose $A$ and $B$ are square injective matrices over a ring $\mathcal{R}$ and the $\mathcal{R}$-modules* $\mathrm{coker}(A)$ *and* $\mathrm{coker}(B)$ *are isomorphic. Then there are identity matrices $I_m, I_n$ and $k \in \mathbb{N}$ and $U, V$ in* $\mathrm{GL}(k,\mathcal{R})$ *such that $U(A \oplus I_m)V = B \oplus I_n$.*

Next, we compile some characterizations of shift equivalence as a theorem. The equivalence of (1), (2) and (3) below is well known. The equivalence of (1) and (4) is what we need for Theorem 2.13. For an $n \times n$ matrix $A$ over a ring $\mathcal{R}$, the $\mathcal{R}[t]$ module $\overline{\mathcal{R}_A}$ is direct limit $\mathcal{R}$-module $\mathcal{R}^n \xrightarrow{A} \mathcal{R}^n \xrightarrow{A} \mathcal{R}^n \xrightarrow{A} \cdots$, with $t$ acting by

$[v,i] \mapsto [v,i+1]$ (inverse to $[v,i] \mapsto [Av,i]$).

For $n \in \mathbb{N}$, $0_n$ and $I_n$ denote the $n \times n$ zero and identity matrices.

For a square matrix $A$ over $\mathcal{R}$, $E(A,\mathcal{R})$ denotes $\mathrm{ElSt}_{\mathcal{R}[t]}(I-tA)$, as in (2.23).

**Theorem 2.12.** *Suppose $A$ and $B$ are square matrices over a ring $\mathcal{R}$. Then the following are equivalent.*

1. *$A$ and $B$ are shift equivalent over $\mathcal{R}$.*

2. *$\overline{\mathcal{R}_A}$ and $\overline{\mathcal{R}_B}$ are isomorphic $\mathcal{R}[t]$ modules.*

3. *$\operatorname{coker}(I-tA)$ and $\operatorname{coker}(I-tB)$ are isomorphic $\mathcal{R}[t]$ modules.*

4. *There are $k,m,n \in \mathbb{N}$ and $U,V$ in $\mathrm{GL}(k,\mathcal{R}[t])$ such that*

   $U\big((I-tA)\oplus I_m\big)V = \big((I-tB)\oplus I_n\big)$ *, i.e.,*

   $U\big((I-t(A\oplus 0_m)\big)V = \big(I-t(B\oplus 0_n)\big)$ .

*If $A$ and $B$ are shift equivalent over $\mathcal{R}$, then $E(A,\mathcal{R}) = E(B,\mathcal{R})$.*

*Proof.* (1) $\iff$ (2) See [3, p.122]. This connection is due to Krieger; the result for $\mathcal{R}=\mathbb{Z}$ was a piece of his introduction of dimension groups to symbolic dynamics [50]. Another proof for the case $\mathcal{R}=\mathbb{Z}$ can be found in [14, 7.5.6–7.5.7].

(2) $\iff$ (3) The map $[v,i] \mapsto [t^i v]$ defines an $\mathcal{R}[t]$-module isomorphism $\overline{\mathcal{R}_A} \to \operatorname{coker}(I-tA)$. This connection was introduced by Kim, Roush and Wagoner [30], for $\mathcal{R}=\mathbb{Z}$.

(4) $\implies$ (3) Clear.

(3) $\implies$ (4) $I-tA$ and $I-tB$ are injective matrices over $\mathcal{R}[t]$, so (4) follows by Theorem 2.11.

Because (1) implies (4), the final claim of the theorem follows from the final claim of Proposition 2.26. □

We let $\sim$ denote $\mathrm{El}(\mathcal{R}[t])$ equivalence.

**Theorem 2.13.** *Let $\mathcal{R}$ be a ring, and $A$ a square matrix over $\mathcal{R}$. The following hold.*

1. *If $B$ is shift equivalent over $\mathcal{R}$ to $A$, then there is a nilpotent matrix $N$ over $\mathcal{R}$ such that $B$ is SSE over $\mathcal{R}$ to the matrix $A \oplus N$.*
2. *For nilpotent matrices $N_1, N_2$ over $\mathcal{R}$, the matrices $A \oplus N_1$ and $A \oplus N_2$ are SSE over $\mathcal{R}$ iff $I - tN_1$ and $I - tN_2$ are the same element in $NK_1(\mathcal{R})/E(A, \mathcal{R})$.*
3. *If $A$ is shift equivalent over $\mathcal{R}$ to a matrix which is nilpotent, invertible or idempotent, then $E(A, \mathcal{R})$ is the trivial group.*

*Proof.* For the proof of (1), suppose $B$ is shift equivalent over $\mathcal{R}$ to $A$. Let $k, m, n, U, V$ be as in (4) of Theorem 2.12. After replacing $A$ with $A \oplus 0_m$ and $B$ with $B \oplus 0_n$ (which is harmless), we have $(I - tB) = U(I - tA)V$ .

Because $\left(\begin{smallmatrix} VU & 0 \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} I-tA & 0 \\ 0 & I \end{smallmatrix}\right) = \left(\begin{smallmatrix} V & 0 \\ 0 & V^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} U(I-tA)V & 0 \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} V^{-1} & 0 \\ 0 & V \end{smallmatrix}\right)$ , we have $I - tB \sim W(I - tA)$, where $W = VU$. Setting $t = 0$, we see $W$ represents an element of $NK_1(\mathcal{R})$. So, for some $j$, after replacing $W$ with $W \oplus I_j$ there exists $N$ nilpotent over $\mathcal{R}$ and $E$ and $F$ elementary over $\mathcal{R}[t]$ such that $EWF = I - tN$. After replacing

$A$ with $A \oplus 0_j$, we have

$$I - tB \sim W(I - tA) \sim (I - tA) \oplus W \sim \begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} I - tA & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & F \end{pmatrix}$$

$$= (I - tA) \oplus (I - tN) = I - t(A \oplus N)\ .$$

Now Theorem 4.6 implies $B$ is strong shift equivalent over $\mathcal{R}$ to $A \oplus N$. This proves (1).

For (2), suppose $N_1, N_2$ are nilpotent matrices over $\mathcal{R}$. By Theorem 4.6, the matrices $A \oplus N_1$ and $A \oplus N_2$ are SSE over $\mathcal{R}$ iff $(I - t(A \oplus N_1)) \sim (I - t(A \oplus N_2))$. For $N$ nilpotent, $(I - t(A \oplus N) \sim (I - tN)(I - tA)$. Therefore $A \oplus N_1$ and $A \oplus N_2$ are SSE over $\mathcal{R}$ iff $(I - tN_1)(I - tA) \sim (I - tN_2)(I - tA)$. By Proposition 2.26, this holds iff $(I - tN_1)^{-1}(I - tN_2) \in \mathrm{ElSt}(I - tA)$ . By Theorem 2.7, this inclusion holds iff $I - tN_1 = I - tN_2$ in $K_1(\mathcal{R}[t])/E(A, \mathcal{R})$ (equivalently, in $NK_1(\mathcal{R})/E(A, \mathcal{R})$). This proves (2).

(3) holds by Theorem 2.7 and the final claim of Theorem 2.12. Note, the nilpotent matrices form the shift equivalence class of the zero matrices. □

**Corollary 2.14.** *Suppose* $\mathrm{NK}_1(\mathcal{R})$ *is trivial (for example, when* $\mathcal{R}$ *is a Noetherian regular ring). Then SE-*$\mathcal{R}$ *implies SSE-*$\mathcal{R}$*.*

Corollary 2.14 answers in the affirmative a question of Wagoner [17, Sec. 9, Problem Number 3]: does SE-$\mathcal{R}$ implies SSE-$\mathcal{R}$ when $\mathcal{R}$ is a commutative regular ring?

Given $\mathcal{R}$ and a square matrix $B$ over $\mathcal{R}$, let $[B]_{SSE}$ denote the SSE-$\mathcal{R}$ class of

$B$ and let $[B]_{SE}$ denote the SE-$\mathcal{R}$ class. For a square matrix $A$ over $\mathcal{R}$, define

$$\mathrm{SSE}/\mathrm{SE}(A,\mathcal{R}) = \{[B]_{SSE} : \ [A]_{SE} = [B]_{SE}\} \ . \tag{2.34}$$

We can now give a short summary of the correspondence provided by Theorem 2.13.

**Theorem 2.15.** *Let $N$ range over nilpotent matrices over $\mathcal{R}$. Then for any square matrix $A$ over $\mathcal{R}$, the map $[I - tN] \to [A \oplus N]_{SSE}$ is a well-defined bijection*

$$NK_1(\mathcal{R})/E(A,\mathcal{R}) \to \mathit{SSE}/\mathit{SE}(A,\mathcal{R})$$

*Equivalently, the map $[N] \to [A \oplus N]_{SSE}$ is a well defined bijection*

$$\mathrm{Nil}_0(\mathcal{R})/E_{\mathrm{Nil}}(A,\mathcal{R}) \to \mathit{SSE}/\mathit{SE}(A,\mathcal{R})$$

*with $E_{\mathrm{Nil}}(A,\mathcal{R}) = \{[N] \in \mathrm{Nil}_0(\mathcal{R}) : [I - tN] \in E(A,\mathcal{R})\}$ .*

Using Theorems 2.12 and 4.6, we record a restatement of Theorem 2.13.

**Theorem 2.16.** *Let $\mathcal{R}$ be a ring. Then the following hold.*

1. *If $A, B$ are square matrices over $\mathcal{R}$ such that the $\mathcal{R}[t]$-modules* $\mathrm{coker}(I - tA)$, $\mathrm{coker}(I - tB)$ *are isomorphic, then there is a nilpotent matrix $N$ over $\mathcal{R}$ such that $I - tB \sim I - t(A \oplus N)$.*

2. *Suppose $N_1, N_2$ are nilpotent matrices over $\mathcal{R}$. Then*

$$I - t(A \oplus N_1) \sim I - t(A \oplus N_2)$$

*iff $[I - tN_1]$ and $[I - tN_2]$ are the same element in $NK_1(\mathcal{R})/E(A,\mathcal{R})$.*

## 2.6 SSE as elementary equivalence

The purpose of this section is to prove Theorem 4.6, our central result for connecting strong shift equivalence and algebraic $K$-theory. To prepare for its statement, we give some definitions.

**Definition 2.35.** Given $A \in t\mathcal{R}[t]$, choose $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $A_1, \dots A_k$ are $n \times n$ matrices over $\mathcal{R}$ such that

$$A = \sum_{i=1}^{k} t^i A_i$$

and define a finite matrix $\mathcal{A}^{\square} = \mathcal{A}^{\square(k,n)}$ over $\mathcal{R}$ by the following block form, in which every block is $n \times n$:

$$\mathcal{A}^{\square} = \begin{pmatrix} A_1 & A_2 & A_3 & \dots & A_{k-2} & A_{k-1} & A_k \\ I & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & I & 0 \end{pmatrix}.$$

In the definition, there is some freedom in the choice of $\mathcal{A}^{\square}$: $k$ can be increased by using zero matrices, and $n$ can be increased by filling additional entries of the $A_i$ with zero. These choices do not affect the SSE-$\mathcal{R}$ class of $\mathcal{A}^{\square}$.

With $\sim$ denoting $\mathrm{El}(\mathcal{R}[t])$ equivalence, recall that for finite matrices $I-A$ and $I-B$, $I-A \sim I-B$ by definition means $(I-A)_{\mathrm{st1}} \sim (I-B)_{\mathrm{st1}}$.

**Theorem 2.17.** *Let $\mathcal{R}$ be a ring. Then there is a bijection between the following sets:*

- *the set of* $\mathrm{El}(\mathcal{R}[t])$ *equivalence classes of square matrices $I-A$ with $A$ over $t\mathcal{R}[t]$*
- *the set of SSE-$\mathcal{R}$ classes of square matrices over $\mathcal{R}$.*

*The map to SSE-$\mathcal{R}$ classes is induced by the map $I-A \mapsto \mathcal{A}^{\square}$. The inverse map (from the set of SSE-$\mathcal{R}$ classes) is induced by the map sending $A$ over $\mathcal{R}$ to the matrix $I-tA$.*

*Proof.* We will first show that when $A$ and $B$ are SSE over $\mathcal{R}$, it follows that the matrices $I-tA$ and $I-tB$ are $\mathrm{El}(\mathcal{R}[t])$ equivalent. It suffices to do this for an elementary strong shift equivalence. Suppose $U, V$ are matrices over $\mathcal{R}$ such that $A = UV$ and $B = VU$. Then (as pointed out by Maller and Shub [26]),

$$\begin{pmatrix} I & 0 \\ V & I \end{pmatrix} \begin{pmatrix} A & U \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & U \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ V & I \end{pmatrix}$$

and therefore

$$\begin{pmatrix} I & 0 \\ V & I \end{pmatrix} \begin{pmatrix} I-tA & -tU \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & -tU \\ 0 & I-tB \end{pmatrix} \begin{pmatrix} I & 0 \\ V & I \end{pmatrix} .$$

Also,

$$\begin{pmatrix} I - tA & -tU \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & -tU \\ 0 & I \end{pmatrix} \begin{pmatrix} I - tA & 0 \\ 0 & I \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} I & -tU \\ 0 & I - tB \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I - tB \end{pmatrix} \begin{pmatrix} I & -tU \\ 0 & I \end{pmatrix} .$$

Therefore $I - tA$ and $I - tB$ are $\mathrm{El}(\mathcal{R}[t])$ equivalent.

Now suppose that $A$ and $B$ are matrices over $t\mathcal{R}[t]$ such that $I - A$ and $I - B$ are $\mathrm{El}(\mathcal{R}[t])$ equivalent. We will show that $\mathcal{A}^\square$ and $\mathcal{B}^\square$ are SSE over $\mathcal{R}$.

There are basic elementary matrices $E_1, \dots, E_j$ and $F_1, \dots, F_k$ , in each of which the single nonzero offdiagonal term has the form $rt^\ell$, with $r \in \mathcal{R}$ and $\ell \geq 0$, such that

$$E_j \cdots E_2 E_1 (I - A) \; = \; (I - B) F_1 F_2 \cdots F_k \ .$$

Choose the block size $n$ for $A^\square$ and $B^\square$ large enough that each $E_i$ and $F_j$ equals $I$ outside the principal submatrix on indices $\{1, \dots, n\} \times \{1, \dots, n\}$. Let $G_i$ denote the image of $E_i$ in $\mathrm{El}(\mathcal{R})$ under the map induced by $t \mapsto 0$. Recursively, for $0 < i \leq j$, given $A_{i-1}$ we will define $A_i$ over $\mathcal{R}[t]$ such that $(A_i)^\square$ is SSE over $\mathcal{R}$ to $(A_{i-1})^\square$ and also

$$\begin{aligned} E_j \cdots E_{i+1} (I - A_i) \; &= \; (I - B)(F_1 F_2 \cdots F_k)(G_1)^{-1} \cdots (G_i)^{-1} \quad \text{if } i < j \\ (I - A_i) \; &= \; (I - B)(F_1 F_2 \cdots F_k)(G_1)^{-1} \cdots (G_i)^{-1} \quad \text{if } i = j \ . \end{aligned} \tag{2.36}$$

There are two cases.

Case 1: The offdiagonal entry of $E_i$ has the form $rt^\ell$ with $\ell > 0$. In this case, define $A_i$ by the equation $I - A_i = E_i(I - A_{i-1})$. By Lemma 2.37, $(A_i)^\square$ is SSE over

$\mathcal{R}$ to $\mathcal{A}^{\square}$. Equation (2.36) holds because $G_i = I$.

Case 2: $E_i$ has all entries in $\mathcal{R}$. Then define $A_i$ over $t\mathcal{R}[t]$ by the equation $I - A_i = E_i(I - A_{i-1})(E_i)^{-1}$. Equation (2.36) holds because $G_i = E_i$, so for this case it remains to check the strong shift equivalence. Let $E_i$ also denote the restriction of $E_i$ to the finite principal submatrix on indices $\{1, \ldots, n\} \times \{1, \ldots, n\}$, define $D$ to be the block diagonal matrix with $k$ diagonal blocks, each equal to $E_i$. Then $A_i^{\square} = D^{-1} A_{i-1}^{\square} D$, and therefore $A_i^{\square}$ is SSE over $\mathcal{R}$ to $A_{i-1}$.

Define $G = G_j \cdots G_2 G_1 \in \mathrm{El}(\mathcal{R})$. From the preceding we have $\mathcal{A}^{\square}$ SSE over $\mathcal{R}$ to $(A_j)^{\square}$, with $I - A_j = (I - B)(F_1 F_2 \cdots F_k) G^{-1}$ and therefore

$$(I - A_j)G = (I - B)(F_1 F_2 \cdots F_k) \ .$$

Let $H_i$ denote the evaluation of $F_i$ at $t = 0$. Repeating the previous procedure, with the role of left and right interchanged, we find $B_k$ with $(B_k)^{\square}$ and $B^{\square}$ SSE over $\mathcal{R}$, and

$$(H_k)^{-1} \cdots (H_2)^{-1}(H_1)^{-1}(I - A_j)G = (I - B_k) \ .$$

Define $H = H_1 H_2 \cdots H_k$. Then $H^{-1}(I - A_j)G = I - B_k$. Evaluating at $t = 0$, we see $H = G$. Then $B_k = G^{-1} A_j G$; as in Case 2, $(A_j)^{\square}$ is SSE over $\mathcal{R}$ to $(B_k)^{\square}$. This finishes the proof (given Lemma 2.37).

□

**Lemma 2.37.** Let $\mathcal{R}$ be a ring. Suppose $A$ and $B$ are matrices over $t\mathcal{R}[t]$; $\ell$ is a positive integer; $E$ is a basic elementary matrix whose nonzero offdiagonal entry is $E(i_0, j_0) = rt^{\ell}$, with $r \in \mathcal{R}$; and $E(I - A) = I - B$ or $(I - A)E = I - B$.

Then the matrices $\mathcal{A}^{\square}$ and $\mathcal{B}^{\square}$ are SSE over $\mathcal{R}$.

*Proof.* Without loss of generality, suppose for notational simplicity that $(i_0, j_0) = (1, 2)$.

We first give a proof assuming that $E(I-A) = I-B$. Let $A = tA_1+\cdots+t^kA_k$, with the $A_i$ over $\mathcal{R}$, and for later notational convenience set $A_i = 0$ if $i > k$. Since $E(A-I) = B-I$, we have $B = EA - E + I = EA - (E-I)I$. Therefore $B = tB_1 + \cdots + t^{k+l}B_{k+\ell}$, with $B_\ell(1,2) \;=\; A_\ell(1,2) - r$, and

$$B_{i+\ell}(1,j) \;=\; A_{i+\ell}(1,j) + rA_i(2,j) \;, \quad 1 \le i \le k \;,$$

and in all other entries $B = A$.

We first consider the case $\ell = 1$. Let $X$ be the $n \times n$ matrix such that $X(1,2) = 1$ and other entries of $X$ are zero. Let $u_i$ be the row vector which is the second row of $A_i$. Let $U_i$ be the $n \times n$ matrix whose first row is $u_i$ and whose other rows are zero. Then the matrix $\mathcal{B}^\square$, in block form with $n \times n$ blocks, is

$$\mathcal{B}^\square = \begin{pmatrix} A_1 - rX & A_2 + rU_1 & A_3 + rU_2 & \dots & A_{k-1} + rU_{k-2} & A_k + rU_{k-1} & rU_k \\ I & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & I & 0 \end{pmatrix} .$$

We we will perform a string of elementary SSEs over $\mathcal{R}$ which will transform $\mathcal{B}^\square$ into $\mathcal{A}^\square$. We use lines within matrices to emphasize block patterns, especially for blocking compatible with a multiplication.

First we perform the column splitting which splits off columns which isolate all entries with coefficient $r$. Letting $e_1$ denote the size $n$ column vector with first entry 1 and other entries zero, we define the $n(k+1) \times n(2k+1)+1$ matrix

$$W = \left(\begin{array}{cccccc|ccc|c} A_1 & A_2 & \dots & A_{k-1} & A_k & 0 & rU_1 & \cdots & rU_k & -re_1 \\ I & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \dots & 0 & I & 0 & 0 & \cdots & 0 & 0 \end{array}\right).$$

and the $(n(2k+1)+1) \times n(k+1)$ matrix

$$M = \left(\begin{array}{cc} I_n & 0 \\ 0 & I_{nk} \\ \hline 0 & I_{nk} \\ \hline e_2 & 0 \end{array}\right)$$

in which $I_j$ as usual means a $j \times j$ identity matrix and $e_2$ is the row vector $(\,0\ 1\ 0\ \cdots\ 0\,)$.

Then $\mathcal{B}^\square = WM$ and we define $B^{(1)} = MW$, SSE over $\mathcal{R}$ to $\mathcal{B}^\square$. In block form,

$$B^{(1)} = \left(\begin{array}{cccccc|ccc|c}
A_1 & A_2 & \dots & A_{k-1} & A_k & 0 & rU_1 & \cdots & rU_k & -re_1 \\
I & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & I & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & I & 0 & 0 & \cdots & 0 & 0 \\
\hline
I & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & I & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & I & 0 & 0 & \cdots & 0 & 0 \\
\hline
u_1 & u_2 & \dots & u_{k-1} & u_k & 0 & 0 & \cdots & 0 & 0
\end{array}\right) .$$

Next we perform a diagonal refactorization of $B^{(1)}$. Define the diagonal matrix $D$ by setting

$$\begin{aligned}
D(t,t) &= 1 \quad \text{if } 1 \le t \le (k+1)n \\
D\big((k+i)n+t,\,(k+i)n+t\big) &= u_i(t) \;\; \text{if } 1 \le i \le k \;\text{ and }\; 1 \le t \le n \\
&= 1 \quad \text{if } t = (2k+1)n+1 \,.
\end{aligned}$$

Define a matrix $X$ which is equal to $B^{(1)}$ except that $X(1,t) = r$ if $(k+1)n+1 \le$

$t \leq (2k+1)n$. Then $B^{(1)} = XD$. Define $B^{(2)} = DX$. In block form,

$$B^{(2)} = \left(\begin{array}{cccccc|ccc|c}
A_1 & A_2 & \dots & A_{k-1} & A_k & 0 & R & \cdots & R & -re_1 \\
I & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & I & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & I & 0 & 0 & \cdots & 0 & 0 \\
\hline
U_1' & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & U_2' & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & U_{k-1}' & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & U_k' & 0 & 0 & \cdots & 0 & 0 \\
\hline
u_1 & u_2 & \dots & u_{k-1} & u_k & 0 & 0 & \cdots & 0 & 0
\end{array}\right)$$

in which every entry of the top row of $R$ is $r$ and the other entries of $R$ are zero, and $U_i'$ denotes the diagonal matrix with $U_i'(t,t) = u_i(t)$, for $1 \leq t \leq n$.

Next, amalgamate the columns $(k+1)n+1, \dots, (2k+1)n$ (the columns through the $R$ blocks) to a single column to form $B^{(3)}$. For this define

$$Y = \left(\begin{array}{ccccccc|c|c}
A_1 & A_2 & A_3 & \cdots & A_{k-1} & A_k & 0 & re_1 & -re_1 \\
I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & \cdots & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & I & 0 & 0 & 0 \\
\hline
U_1' & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & U_2' & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & U_3' & \cdots & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & U_{k-1}' & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & U_k' & 0 & 0 & 0 \\
\hline
u_1 & u_2 & u_3 & \cdots & u_{k-1} & u_k & 0 & 0 & 0
\end{array}\right) \quad \text{and}$$

$$Z = \left(\begin{array}{c|c|c}
I_{(k+1)n} & 0 & 0 \\
\hline
0 & 1\cdots 1 & 0 \\
\hline
0 & 0 & 1
\end{array}\right)$$

in which the central block of $Z$ is a row vector of size $kn$ with every entry 1. Then

$B^{(2)} = YZ$ and we define $B^{(3)} = ZY$. In block form,

$$B^{(3)} = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_{k-1} & A_k & 0 & re_1 & -re_1 \\ I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & I & 0 & 0 & 0 \\ u_1 & u_2 & u_3 & \cdots & u_{k-1} & u_k & 0 & 0 & 0 \\ u_1 & u_2 & u_3 & \cdots & u_{k-1} & u_k & 0 & 0 & 0 \end{pmatrix}.$$

Next we similarly amalgamate the last two rows, to obtain the matrix

$$B^{(4)} = \left(\begin{array}{cccccc|c|c} A_1 & A_2 & A_3 & \cdots & A_{k-1} & A_k & 0 & 0 \\ I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 & 0 & 0 \\ I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & I & 0 & 0 \\ \hline u_1 & u_2 & u_3 & \cdots & u_{k-1} & u_k & 0 & 0 \end{array}\right).$$

This matrix is a zero extension of $\mathcal{A}^{\square}$ and therefore is SSE over $\mathcal{R}$ to $\mathcal{A}^{\square}$ (see Proposition 6.5). This finishes the proof in the case $\ell = 1$ that the matrices $\mathcal{A}^{\square}$ and $\mathcal{B}^{\square}$ are SSE over $\mathcal{R}$.

The proof for the case $\ell > 1$ is very similar. We will discuss it for the case $\ell = 3$, from which the general argument should be clear. For $\ell = 3$, with the same notation as in the case $\ell = 1$, and recalling $A_i = 0$ if $i > k$, we have

$$\mathcal{B}^{\square} = \begin{pmatrix} A_1 & A_2 & A_3 - rX & A_4 + rU_1 & \cdots & A_{k+1} + rU_{k-2} & A_{k+2} + rU_{k-1} & A_{k+3} + rU_k \\ I & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & I & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & I & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & I & 0 \end{pmatrix}.$$

As in the case $\ell = 1$, we split columns to isolate the terms involving $r$. The resulting matrix $B^{(1)}$ here has a form involving a shift of the $\ell = 1$ form in the new rows:

$$B^{(1)} = \left(\begin{array}{cccccccc|ccc|c}
A_1 & A_2 & A_3 & A_4 & \cdots & A_{k+1} & A_{k+2} & 0 & rU_1 & \cdots & rU_k & -re_1 \\
I & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & I & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & I & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & I & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & I & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & I & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & u_1 & u_2 & \cdots & u_{k-1} & u_k & 0 & 0 & \cdots & 0 & 0
\end{array}\right).$$

From here the argument proceeds as in the case $\ell = 1$, through slightly different matrices,

$$
B^{(2)} = \left(\begin{array}{cccccccc|ccc|c}
A_1 & A_2 & A_3 & A_4 & \cdots & A_{k+1} & A_{k+2} & 0 & R & \cdots & R & -re_1 \\
I & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & I & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & I & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & U_1' & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & U_2' & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & U_{k-1}' & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & U_k' & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & u_1 & u_2 & \cdots & u_{k-1} & u_k & 0 & 0 & \cdots & 0 & 0
\end{array}\right)
$$

and

$$B^{(3)} = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & \cdots & A_{k+1} & A_{k+2} & 0 & re_1 & -re_1 \\ I & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & I & 0 & 0 & 0 \\ 0 & 0 & u_1 & u_2 & \cdots & u_{k-1} & u_k & 0 & 0 & 0 \\ 0 & 0 & u_1 & u_2 & \cdots & u_{k-1} & u_k & 0 & 0 & 0 \end{pmatrix}.$$

This completes our proof that $\mathcal{A}^\square$ and $\mathcal{B}^\square$ are SSE over $\mathcal{R}$ in the case $E(I-A) = I - B$.

Now suppose $(I-A)E = I - B$. In place of $\mathcal{A}^\square$, we consider a matrix form corresponding to a role reversal for rows and columns:

$$A^{\mathrm{col}} = \begin{pmatrix} A_1 & I & 0 & \cdots & 0 & 0 & 0 \\ A_2 & 0 & I & \cdots & 0 & 0 & 0 \\ A_3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{k-2} & 0 & 0 & \cdots & 0 & I & 0 \\ A_{k-1} & 0 & 0 & \cdots & 0 & 0 & I \\ A_k & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

With the roles of row and column reversed, the arguments we've given show that $A^{\mathrm{col}}$ and $B^{\mathrm{col}}$ are SSE over $\mathcal{R}$. What remains is to see that $A^{\mathrm{col}}$ and $\mathcal{A}^\square$ are SSE over $\mathcal{R}$. For this we define a matrix $A'$ with the block form

$$A' = \left(\begin{array}{cc|cc|cc|c|cc|cc}
A_1 & A_2 & A_3 & 0 & A_4 & 0 & \cdots & A_{k-1} & 0 & A_k & 0 \\
I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & I_n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & I_{2n} & \cdots & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\hline
\cdots & \cdots & \multicolumn{2}{c|}{\cdots} & \multicolumn{2}{c|}{\cdots} & \cdots & \multicolumn{2}{c|}{\cdots} & \multicolumn{2}{c}{\cdots} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & I_{(k-3)n} & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & I_{(k-2)n} \\
I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{array}\right).$$

In the display of $A'$ above and next, a block $I$ without subscript is $I_n$.

For example, if $k = 4$ then

$$A' = \left(\begin{array}{cc|cc|ccc} A_1 & A_2 & A_3 & 0 & A_4 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ I & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Three steps remain.

First, the matrix $A'$ is SSE over $\mathcal{R}$ to $\mathcal{A}^{\square}$ by a string of $k-2$ block row amalgamations. Beginning with $A' = A'_0$: amalgamate to block row 2 the block rows with $I$ in block column 1 to form $A'_1$. From the resulting matrix, amalgamate to block row 3 the rows with $I$ in column 2, to form $A'_2$. Etc. The last block row amalgamation produces $\mathcal{A}^{\square}$. For example, with $A'$ above for $k = 4$ and

$$X = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & I & 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} A_1 & A_2 & A_3 & 0 & A_4 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}$$

we have $A' = A'_0 = XY$ and

$$A'_1 = YX = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & I & 0 & 0 & 0 \end{pmatrix} .$$

The next step produces $A'_2 = \mathcal{A}^{\Box}$.

Second, the matrix $A'$ is conjugate to the matrix $A^*$ obtained from $A'$ by (i) replacing in block row 1 the blocks $A_j$, $2 \leq j \leq k$, with the identity block $I_n$ and (ii) replacing the $I$ blocks in block column 1 with $A_2, \dots, A_k$ (with $A_j$ appearing above $A_{j+1}$, $1 \leq i < k$). An SSE from $A'$ to $A^*$ is achieved by a string of diagonal refactorizations of the blocks $A_j$. For example, in the display for $k = 4$, let $X$ be the matrix obtained from $A'$ by replacing the $A_2$ block with $I$. Let $D$ be the block diagonal matrix with block indices matching those of $A'$, and with $D = A_2$ in the second diagonal block and $D = I$ otherwise. Then $XD = A'$ and $DX$ has $A_2$ occupying the $2,1$ block as desired. To move $A_j$ to its target position in the first block column takes $j - 1$ moves of this type.

Third and last, the matrix $A^*$ is SSE over $\mathcal{R}$ to the matrix $A^{\text{col}}$ by a string of block column amalgamations, just as $A'$ is SSE over $\mathcal{R}$ to $\mathcal{A}^{\Box}$ by a string of block row amalgamations.

This finishes the proof of the lemma. $\Box$

We record a corollary of Theorem 4.6.

**Corollary 2.18.** *Suppose $\mathcal{R}$ is a ring, and suppose $P$ and $Q$ are square matrices over $\mathcal{R}[t]$. Suppose $A'$ and $B'$ are matrices over $\mathcal{R}$ such that $P$ and $Q$ are $\mathrm{El}(\mathcal{R}[t])$ equivalent (respectively) to $I - tA'$ and $I - tB'$. Then the following are equivalent:*

1. *$A'$ and $B'$ are SSE over $\mathcal{R}$.*

2. *$P$ and $Q$ are $\mathrm{El}(\mathcal{R})[t]$ equivalent.*

## 2.7 SSE and $\mathrm{Nil}_0(\mathcal{R})$

Nilpotent matrices $N, N'$ over $\mathcal{R}$ represent the same element of $\mathrm{Nil}_0(\mathcal{R})$ if and only if $I - tN$ and $I - tN'$ represent the same element of $NK_1(\mathcal{R})$. It therefore follows from Theorem 4.6 that there is another characterization of when nilpotent matrices $N, N'$ represent the same element of $\mathrm{Nil}_0(\mathcal{R})$:

**Theorem 2.19.** *Suppose $N$ and $N'$ are nilpotent matrices over a ring $\mathcal{R}$. Then the following are equivalent.*

1. *$[N] = [N']$ in $\mathrm{Nil}_0(\mathcal{R})$.*

2. *$N$ and $N'$ are SSE over $\mathcal{R}$.*

(There is also a shorter proof of Theorem 2.19, avoiding Theorem 4.6, which we forego.) Consequently, we can think of Theorem 4.6 as a generalization of the correspondence $\mathrm{Nil}_0(\mathcal{R}) \to \mathrm{NK}_1(\mathcal{R})$, from nilpotent matrices to arbitrary matrices.

*Remark* 2.38. Theorem 4.6 is an alternate ingredient for a proof that $\mathrm{Nil}_0(\mathcal{R})$ and $NK_1(\mathcal{R})$ are isomorphic. If the matrix $A$ in $\mathcal{M}(\mathcal{R})$ is nilpotent, then the map

$\beta\colon A \mapsto I - tA$ is the standard map inducing the group isomorphism $\mathrm{Nil}_0(\mathcal{R}) \to \mathrm{NK}_1(\mathcal{R})$. It is straightforward to check that $\beta$ induces a well defined homomorphism $\mathrm{Nil}_0(\mathcal{R}) \to \mathrm{NK}_1(\mathcal{R})$, which is surjective on account of the Higman trick (see [19] Proposition 3.5.3, or [18] Theorem 3.2.22). The more difficult part of the proof is to show that this epimorphism is injective. For example, Weibel proves this with a sophisticated composition of maps (see [19], Section III.3.5). Rosenberg approaches this by defining a map inducing the inverse, but (he agrees that) the proof [18, p.150] that the map is well defined is incomplete. The map of Theorem 4.6 restricts to define an inverse to the standard epimorphism $\mathrm{Nil}_0(\mathcal{R}) \to \mathrm{NK}_1(\mathcal{R})$, and therefore gives an alternate proof for this step, in the spirit of Rosenberg's approach. It also identifies the elements of $\mathrm{Nil}_0(\mathcal{R})$ as SSE-$\mathcal{R}$ classes.

# Chapter 3: Finite group extensions of shifts of finite type

## 3.1 Introduction

One part of the celebrated paper [51] of Livšic shows that for certain hyperbolic dynamical systems $T : X \to X$, if the restrictions of Hölder functions $f$ and $g$ to the periodic points are cohomologous as point set maps (i.e. ignoring topology), then they are Hölder cohomologous in $(X, T)$ — i.e., $f = g + r \circ T - r$, with the transfer function $r$ being Hölder continuous. (For an excellent introdiction to the Livšic theory and to cocycles in dynamical systems, see [52].) The proof of Livšic works for functions into a metrizable abelian group. This result was generalized to nonabelian groups for shifts of finite type by Parry (see Remark 3.26) and Schmidt [4, 5], and to more sophisticated systems by various authors (e.g. [4–7]).

Parry posed a bold related question in the case $G$ is finite abelian. For $(X, T)$ a mixing SFT and $f : X \to G$, a suitable dynamical zeta function $\zeta_f$ encodes for all $n, g$ the number of periodic orbits of size $n$ and weight $g$. Then $\zeta_f = \zeta_g$ if and only if there is a bijection $\beta : \mathrm{Per}(X) \to \mathrm{Per}(X)$ such that $f \circ \beta$ and $g$ are cohomologous as point set maps. Parry asked, for $f : X \to G$ continuous and $G$ a finite abelian group: does the set of continuous $g : X \to G$ with $\zeta_g = \zeta_f$ contain only finitely many continuous cohomology classes? Parry's question probed not only a possible direction for extending the Livšic result, but also the strength of conjugacy invariants for mixing SFTs and their group extensions. (The classification of cohomology classes of functions from $X$ into a group is a version of the classification of group extensions of a system $(X, T)$.)

We will show that for many groups $G$ (the finite groups $G$ with $NK_1(\mathbb{Z}G) \neq 0$), the answer to Parry's question is negative for *every* nontrivial dynamical zeta function. The ingredients for this are the following.

1. Generalizing the Williams' theory for SFTs, Parry showed that any $G$-extension of an SFT $(X, S)$ can be presented by a square matrix $A$ over $\mathbb{Z}_+G$, and two such group extensions are isomorphic if and only if their presenting matrices are strong shift equivalent (SSE) over the positive semiring $\mathbb{Z}_+G$ of the integral group ring $\mathbb{Z}G$. The dynamical zeta function, with coefficient ring $\mathbb{Z}G$, is then $\zeta(z) = (\det(I - zA))^{-1}$. Parry's theory, which he never published, is presented in [32] (in Appendix 3.7, we correct an error in the presentation in [32]). [1]

2. By Theorem 4.5, taken from [45], for any ring $\mathcal{R}$ and shift equivalence (SE) class $\mathcal{C}$ of matrices over $\mathcal{R}$, the collection of SSE classes over $\mathcal{R}$ of matrices in $\mathcal{C}$ is in bijective correspondence with the group $NK_1(\mathcal{R})$ of algebraic K-theory. If $NK_1(\mathcal{R})$ is not trivial, then it is not finitely generated as a group [19, 20]. We give more background on $NK_1(\mathcal{R})$ in Appendix 3.9, and give some concrete

[1] The algebraic invariants here over $\mathbb{Z}$ (shift and strong shift equivalence, $\det(I - tA)$), are parallelled in the study of shifts of finite type with Markov measure, where a finitely generated abelian group appears in place of the finite group $G$ [53, 54], and positivity issues around $\det(I - tA)$ and shift equivalence become more analytic and formidable [55].

examples in Appendix 3.9.

3. In this paper, given $NK_1(\mathbb{Z}G)$ nontrivial, we construct, for any nontrivial mixing SFT $(X, S)$, infinitely many $G$-extensions of $(X, S)$ defined by matrices which pairwise are SE over $\mathbb{Z}_+G$ but are not SSE over $\mathbb{Z}G$ (and hence are not SSE over $\mathbb{Z}_+G$). Consequently, these extensions pairwise are eventually conjugate; are not conjugate; and have the same isomorphism class of conjugacy classes (in the abelian case, this means they have the same dynamical zeta function). The construction arguments, carried out in Section 3.5, use constructive tools available in the polynomial matrix setting.

In Section 3.4, we discuss Parry's question in more detail, and we use the structure of shift equivalence of matrices over $\mathbb{Z}G$ to address and clarify some other cases of Parry's question (Sec. 3.4). We show that for every nontrivial finite group $G$, there is an infinite collection of matrices which are not SE-$\mathbb{Z}G$ and which can be realized in mixing extensions of SFTs with the same periodic data. Consequently, for every nontrivial finite abelian group $G$, there is a dynamical zeta function compatible with infinitely many SE-$\mathbb{Z}G$ classes which can be realized in mixing extensions of SFTs. On the other hand, we give a class of mixing examples for which the dynamical zeta function determines the SE-$\mathbb{Z}G$ class (regardless of $\mathrm{NK}_1(\mathbb{Z}G)$). For such a class, known invariants do not provide a negative answer to Parry's question. In no nontrivial case do known constructions provide a positive answer to Parry's question.

One purpose of this paper is to summarize and extend our understanding of the algebraic invariants for and approaches to mixing finite group extensions of shifts of finite type (which we need anyway for Parry's question). (In particular, for not necessarily abelian finite groups $G$, we give complete and computable invariants for the periodic data of the $G$ extension of a shift of finite type.) There are two parallel formulations for this. One involves SSE of matrices over $\mathbb{Z}G$ (Section 3.2). The other formulation is in terms of the "positive K-theory" of polynomial matrix presentations (Section 3.3). In Appendix 3.8, we work out results involving primitivity (some of which we need for proofs) and shift equivalence to extend the theory parallel to the theory over $\mathbb{Z}$. In Appendix 3.7 we review the basic connection of matrices over $\mathbb{Z}_+G$ to $G$-extensions, and correct a mistake in [32]. (The mistake is only that the defining matrix should be associated to a left action of $G$, not a right action.) Some open problems are listed in Section 7.

## 3.2 Finite group extensions of SFTs via matrices over $\mathbb{Z}G$

In this section we give basic definitions for finite group extensions; describe the presentation of group extensions of SFT by matrices over $\mathbb{Z}_+G$; and describe algebraic invariants of defining matrices which correspond to invariants of the group extensions. Cocycles and the group extension construction are an important tool much more generally in dynamics (topological, measurable and smooth), but for simplicity, we restrict definitions to our special case. We recommend [52] for an introduction to cocycles in dynamics; [32] is a reference with proofs adapted to some of the items below, as indicated by references.

**Standing assumption.** Unless indicated otherwise, from here $G$ denotes a finite group. All $G$ actions are assumed to be continuous and free unless indicated.

**Basic definitions [32].** Let a pair $(X, S)$ represent a homeomorphism $S : X \to X$. We will be interested in only two cases: either $(X, S)$ is a shift of finite type, or it is a countable union of finite orbits, with the discrete topology (i.e., we neglect topology). A *group extension* of $(X, S)$ by $G$ is a pair $(Y, T)$ together with a continuous map $\pi : (Y, T) \to (X, S)$ such that $S\pi = \pi T$; two points have the same image under $\pi$ if and only if they are in the same $G$-orbit; and $\pi$ is a covering map (for each point $x$ of $X$, there is a neigborhood $V$ such that there are $|G|$ disjoint neighborhoods in $Y$ such that the restriction of $\pi$ to each is a homeomorphism onto $V$). If $(X, S)$ is SFT, then a $G$ extension of $(X, S)$ is a free $G$-SFT, i.e. an SFT $(X, S)$ together with a continuous free action of $G$ which commutes with the shift. We will always take $G$ acting *from the left*, for a correct matrix correspondence in the case $G$ is nonabelian – see Appendix 3.7 for an explanation, which corrects the choice "from the right"in [32].

Two $G$ extensions $(Y_1, T_1), (Y_2, T_2)$ are *conjugate*, or *isomorphic*, if there is a homeomorphism $\phi : Y_1 \to Y_2$ such that $\phi T_1 = T_2 \phi$ and $\phi(gy) = g\phi(y)$ for all $y \in Y_1$. Equivalently, they are isomorphic as $G$-SFTs. A $G$ extension of $(X, S)$ may be constructed from a continuous function $\tau : X \to G$ (a *skewing function*) as follows. Let $Y = X \times G$ and define $T : Y \to Y$ by the rule $(x, g) \mapsto (S(x), g\tau(x))$, with $\pi : X \times G \to X$ the obvious map $(x, g) \mapsto x$. Every $G$-extension of an SFT is isomorphic to one constructed in this way, and for brevity we may refer to such a group extension as $(X, S, \tau)$.

We say $G$-extensions $(X_1, S_1, \tau_1)$ and $(X_2, S_2, \tau_2)$ are *eventually conjugate* if for all but finitely many $n > 0$ the $G$-extensions $(X_1, (S_1)^n, \tau_1)$ and $(X_2, (S_2)^n, \tau_2)$ are conjugate.

In a system $(X, S)$, continuous functions $\tau_1$ and $\tau_2$ from $X$ to $G$ are *cohomologous* if there is a continuous function $\gamma : X \to G$ such that for all $x$, $\tau_1(x) = (\gamma(x))^{-1}(\tau_2(x))\gamma(Sx)$ in the group $G$. For $G$-extensions $(X_1, S_1, \tau_1)$ and $(X_2, S_2, \tau_2)$, the following are equivalent:

1. The two $G$-extensions are isomorphic.
2. There is a homeomorphism $\phi : X_1 \to X_2$ such that $\phi S_1 = S_2\phi$ (i.e. $\phi$ is a topological conjugacy) and the functions $\tau_2 \circ \phi$ and $\tau_1$ are cohomologous in $(X_1, S_1)$.

A *mixing* $G$-extension of $(X, S)$ is a $G$-extension $(Y, T)$ of $(X, S)$ such that $(Y, T)$ is topologically mixing. This is distinctly a stronger assumption than the assumption that $(X, S)$ is mixing. The mixing $G$-extensions are the fundamental, central case. (The papers [56,57] of Adler, Kitchens and Marcus describe invariants with which the classification of some $G$ extensions of SFTs can be reduced to this central case.)

**Presentation by matrices over $\mathbb{Z}_+G$ [32].** Suppose $A$ is a square matrix with entries in $\mathbb{Z}_+G$. Then $A$ may be viewed as the adjacency matrix of a labeled directed graph, with adjacency matrix $\overline{A}$ defining an edge SFT $(X, S)$, by setting

$$\tau(x) \;=\; \text{the label of the edge } x_0 \;. \tag{3.1}$$

Then $(X, S, \tau)$ is a group extension of the SFT $(X, S)$. Every group extension of an SFT is isomorphic to one of this type.

**Mixing.** For an element $x = \sum_g n_g g$ of $\mathbb{Z}G$, we write $x \gg 0$ if $n_g > 0$ for every $g$, and say $x$ is $G$-positive. For a matrix $A$ over $\mathbb{Z}G$, $A \gg 0$ means every entry is $\gg 0$. We define a $G$-primitive matrix to be a square matrix over $\mathbb{Z}_+G$ such that $A^n \gg 0$ for some $n > 0$.

A nonzero square matrix $A$ contains a maximum principal submatrix with no zero row and no zero column; this is the *nondegenerate core* of $A$. For a property $P$, a matrix $A$ is *essentially* $P$ if its nondegenerate core is $P$. A matrix $A$ over $\mathbb{Z}_+G$ defines a mixing $G$-extension if and only if it is essentially $G$-primitive (Proposition 3.8).

NOTE: The $\mathbb{Z}_+$ matrix $\overline{A}$ being primitive does not guarantee that $A$ is primitive. (E.g., $A = (e + e)$ over $\mathbb{Z}G$ with $G = \mathbb{Z}/2\mathbb{Z}$.)

**Conjugacy and eventual conjugacy.** $G$-extensions of SFTs presented by matrices $A, B$ over $\mathbb{Z}_+G$ are conjugate if and only if the matrices $A, B$ are strong shift equivalent (SSE) over $\mathbb{Z}_+G$. This theory, due to Parry and never published by him, is presented in [32]. By Proposition 3.50, these $G$-extensions are eventually conjugate if and only if $A, B$ are shift equivalent (SE) over $\mathbb{Z}_+G$. By Proposition 3.51, two $G$-primitive matrices are SE over $\mathbb{Z}_+G$ if and only if they are SE over $\mathbb{Z}G$.

**Refinement of SE-$\mathbb{Z}G$ by SSE-$\mathbb{Z}G$.** For any ring $\mathcal{R}$, the refinement of SE-$\mathcal{R}$ by SSE-$\mathcal{R}$ is captured by the group $\mathrm{NK}_1(\mathcal{R})$ of algebraic K-theory, as follows.

**Theorem 3.1.** *[45] Suppose $A$ is a square matrix over a ring $\mathcal{R}$.*

1. *If $B$ is SE over $\mathcal{R}$ to $A$, then there is a nilpotent matrix $N$ over $\mathcal{R}$ such that $B$ is SSE over $\mathcal{R}$ to the matrix* $A \oplus N = \begin{pmatrix} A & 0 \\ 0 & N \end{pmatrix}$.

2. *The map $[I - tN] \to [A \oplus N]_{SSE}$ induces a bijection from* $\mathrm{NK}_1(\mathcal{R})$ *to the set of SSE classes of matrices over $\mathcal{R}$ which are in the SE-$\mathcal{R}$ class of $A$.*

For more on $\mathrm{NK}_1$, see Appendix 3.9.

**Periodic data and trace series.** We consider $G$-extensions $(X, S, \tau)$ such that $(X, S)$ has only finitely many orbits of size $n$, and formulate "periodic data" which give a complete invariant of isomorphism for the group extension obtained by restriction of $S$ and $\tau$ to the periodic points of $S$, with the discrete topology. (Caveat: in the context of a Livšic type theorem, "periodic data" may refer to the cohomology class of the restriction of $\tau$ to the periodic points, with discrete topology [7]. Our series definition (3.3) is equivalent for the case we consider, being a complete invariant for that class.)

**Definition 3.2.** For $g \in G$, let $\kappa(g)$ denote the conjugacy class of $g$ in $G$ ($= \{g\}$ if $G$ is abelian). Let $\mathbb{Z}\mathrm{Conj}G$ denote the free abelian group with generators the conjugacy classes of $G$. We also let $\kappa$ denote the induced group homomorphism $\mathbb{Z}G \to \mathbb{Z}\mathrm{Conj}G$ given by $\sum n_g g \mapsto \sum n_g \kappa(g)$. We use $\kappa$ similarly for other induced maps.

If $(X_1, S_1, \tau_1)$ is a $G$ extension and $x \in \mathrm{Fix}(S^n)$, set $w(x) = \tau(x)\tau(Sx)\ldots\tau(S^{n-1}x)$ and $\kappa_n(x) = \kappa(w(x))$. If a topological conjugacy $\phi : X_1 \to X_2$ sends $\tau_1$ to a function cohomologous to $\tau_2$, and $x \in \mathrm{Fix}(S^n)$, then $\kappa_n(x) = \kappa_n(\phi(x))$. Given a $G$-extension of $(X, S)$ defined by $\tau$ and a conjugacy class $c$ from $G$, define the *periodic data* to be the formal power series with coefficients in $\mathbb{Z}\mathrm{Conj}G$,

$$P_\tau = \sum_{n=1}^{\infty} \Big( \sum_{x \in \mathrm{Fix}(S^n)} \kappa\big(\tau(x)\tau(Sx)\ldots\tau(S^{n-1}x)\big)\Big) t^n \ . \tag{3.3}$$

Then for $G$ extensions $(X_1, S_1, \tau_1)$ and $(X_2, S_2, \tau_2)$, a necessary and sufficient condition for isomorphism of the $G$ extensions obtained by restriction to their periodic points (neglecting topology) is that $P_{\tau_1} = P_{\tau_2}$.

**Definition 3.4.** Let $A$ be a square matrix over a ring. The *trace series* of $A$ is

$$\mathcal{T}_A = \sum_{n=1}^{\infty} \mathrm{tr}(A^n) t^n \ . \tag{3.5}$$

For $A$ a matrix over $\mathbb{Z}G$, the *conjugacy class trace series* of $A$ is

$$\kappa\mathcal{T}_A = \sum_{n=1}^{\infty} \kappa\big(\mathrm{tr}(A^n)\big) t^n \ . \tag{3.6}$$

The trace series of $A$ and $B$ are *conjugate if* $\kappa\mathcal{T}_A = \kappa\mathcal{T}_B$.

We relate $\kappa\mathcal{T}_A$ to existing $K$-theory invariants [58] in Proposition 3.19. If the extension $(X, S, \tau)$ is defined by a matrix $A$ over $\mathbb{Z}_+G$, then

$$P_\tau = \mathcal{T}_A \ . \tag{3.7}$$

**Periodic data for G abelian.** If $G$ is abelian, we identify $\kappa(g)$ with $g \in \mathbb{Z}G$. Then the periodic data $P_\tau$ for the extension $(X, S, \tau)$ is encoded by the usual dynamical zeta function, taken with coefficients in $\mathbb{Z}G$,

$$\zeta_\tau(z) = \exp\Big( \sum_{n=1}^{\infty} \sum_{x : S^n x = x} \tau(x)\tau(Sx)\cdots\tau(S^{n-1}x)\frac{z^n}{n}\Big) \ .$$

When $\tau : X \to G$ is constructed from a matrix $A$ over $\mathbb{Z}_+G$ as above,

$$\zeta_\tau(t) \;=\; \exp\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}(A^n)t^n \;=\; (\det(I-tA))^{-1} \tag{3.8}$$

and $\det(I-tA)$ is a complete invariant for the periodic data. (Here, $\zeta_\tau$ is an example of a dynamical zeta function. There is a huge literature using variants of such functions; one survey for nonexperts is [59].)

**Periodic data for general** $G$**.** Suppose $A$ has entries in $\mathbb{Z}_+G$ where $G$ need not be abelian. The usual polynomial $\det(I-tA)$ need not be well defined. Nevertheless, by Proposition 3.44, the finite sequence $(\kappa(\mathrm{tr}(A^k))_{1\le k\le mn}$ determines all of $\kappa\mathcal{T}_A$, and the sequence $(\kappa(\mathrm{tr}(A^k))_{1\le k<\infty}$ satisfies a readily computed recursion relation with coefficients in $\mathbb{Z}$. A connection of $\kappa\mathcal{T}_A$ and K-theory is described in Proposition 3.19.

**Periodic data, SE and SSE.** If $A, B$ are SSE over $\mathbb{Z}G$, then $\kappa\mathcal{T}_A = \kappa\mathcal{T}_B$ (Proposition 3.44). If $G$ is a finite abelian group, then $\det(I-tA)$ is an invariant of SE over $\mathbb{Z}G$, as follows. With $B$ SE over $\mathbb{Z}G$ to $A$, by Theorem 4.5 there exists a nilpotent matrix $N$ such that $A\oplus N$ is SSE over $\mathbb{Z}G$ to $B$, and then

$$\det(I-tB) = \det(I-tA)\det(I-tN) = \det(I-tA)$$

with the second equality holding by Proposition 3.52.

For $G$ not abelian, $\mathbb{Z}G$ might contain nonzero nilpotent elements (for example $\mathbb{Z}[D_4]$, where $D_4$ is the dihedral group of order 4, contains nilpotent elements), and in this case the periodic data will not be invariant under SE over $\mathbb{Z}G$. In any case, if $A$ and $B$ are SE over $\mathbb{Z}G$ with lag $\ell$, then $\kappa(\mathrm{tr}(A^k)) = \kappa(\mathrm{tr}(B^k))$ for all $k \ge \ell$, and then $\kappa\mathcal{T}_A = \kappa\mathcal{T}_B$ if and only if $\kappa(\mathrm{tr}(A^k)) = \kappa(\mathrm{tr}(B^k))$ for all $k < \ell$.

**Flow equivalence.** Complete invariants of $G$-equivariant flow equivalence for $G$-SFTs are known in terms of algebraic invariants associated to a presenting $\mathbb{Z}_+G$ matrix $A$ (see [32] for the case $\overline{A}$ primitive and [60] for the general case).

### 3.3 Finite group extensions of SFTs via matrices over $\mathbb{Z}G[t]$

Invariants of group extensions of SFTs can be developed via matrices over $\mathbb{Z}_+G$ with the SSE/SE approach, or via matrices with entries from the polynomial ring $\mathbb{Z}_+G[t]$ with the "positive K-theory" approach of [27, 28]). In this section we recall and develop what we need of the positive K-theory for constructions, and summarize algebraic invariants in this setting.

In this paper, we formulate positive equivalence in terms of finite matrices. The equivalent infinite matrix formulation of positive equivalence described later in this section is used in [27, 28]. Other formulations vary a bit among [28], [27] and the present paper, but they are equivalent where they overlap. The paper [28] is written for matrices over $\mathbb{Z}$ and $\mathbb{Z}_+$, outside of Section 7, which address matrices over integral group rings.

**Positive equivalence.**

Let $R$ be a ring (always assumed to contain 1). A *basic elementary matrix* over $R$ is a square matrix over $R$ equal to the identity except perhaps in a single offdiagonal entry.

Below, $0_n$ is the $n \times n$ zero matrix, $I_n$ is the $n \times n$ identity matrix, and $0, I$ represent zero, identity matrices of appropriate sizes.

Let $\mathcal{M}$ be a set of square matrices $I - A$ over $R$ such that

$$I - A \in \mathcal{M} \implies I - (A \oplus 0_n) \in \mathcal{M} \quad , \qquad \text{for all } n > 0 \quad .$$

Let $\mathcal{S}$ be a subset of $R$ containing zero and one. A *basic elementary equivalence over $\mathcal{S}$ in $\mathcal{M}$* is an equivalence of the form $I - A \mapsto U(I - A) = I - B$ or $I - A \mapsto (I - A)U = I - B$ such that $U$ is a basic elementary matrix, and both $I - A$ and $I - B$ are in $\mathcal{M}$. An equivalence $I - A \mapsto U(I - A)V = I - B$ is an *elementary equivalence over $\mathcal{S}$ in $\mathcal{M}$* if for some $k$, $(U \oplus I_k, V \oplus I_k) : I - (A \oplus I_k) \to I - (B \oplus I_k)$ is a composition of basic elementary equivalences over $\mathcal{S}$ in $\mathcal{M}$. We say square matrices $I - A, I - B$ are *elementary equivalent over $\mathcal{S}$ in $\mathcal{M}$* if there exist $j, k$ such that there is an elementary equivalence over $\mathcal{S}$ in $\mathcal{M}$ from $I - (A \oplus I_j)$ to $I - (B \oplus I_k)$.

**Definition 3.9.** Suppose $R$ is an ordered ring with $R_+$ containing 0 and 1. A square matrix $A$ over $R_+[t]$ has the NZC property if for all $n \geq 0$, every diagonal entry of $A^n$ has constant term zero. NZC($R_+[t]$) is the set of square matrices $A$ over $R_+[t]$ having the NZC property.

For example, the matrix $\left(\begin{smallmatrix} t & 3+t^3 \\ 2t^5 & t \end{smallmatrix}\right)$ is in NZC($Z_+[t]$); the matrix $\left(\begin{smallmatrix} t & 3+t^3 \\ 1+2t & t \end{smallmatrix}\right)$ is not. The square matrices over $tR_+[t]$ are contained in NZC($R_+[t]$).

**Definition 3.10.** Suppose $R$ is an ordered ring with $R_+$ containing 0 and 1. With respect to this ordered ring, two matrices are *positive equivalent* if they are elementary equivalent over $R_+$ in $\mathcal{M}$, where $\mathcal{M}$ is the set of square matrices of the form $I - A$ with $A$ in NZC($R_+$).

In this paper, positive equivalent without modifiers means positive equivalent with respect to $R = \mathbb{Z}G[t]$ and $R_+ = \mathbb{Z}_+G[t]$.

**Positive equivalence and SSE.** The next result is a trivial corollary of [28, Theorem 7.2], but it takes a little space to explain why this is so.

**Theorem 3.2.** *[28, Theorem 7.2] Let $G$ be a group and $\mathbb{Z}G$ its integral group ring. Let $A, B$ be matrices in $\mathrm{NZC}(\mathbb{Z}_+G[t])$ and let $A^\diamond, B^\diamond$ be square matrices over $\mathbb{Z}_+G$ such that $I-A$ and $I-B$ are (respectively) positive equivalent to $I-tA^\diamond$ and $I-tB^\diamond$. Then the following are equivalent.*

1. *$A^\diamond$ and $B^\diamond$ are SSE over $\mathbb{Z}_+G$.*

2. *$I-A$ and $I-B$ are positive equivalent.*

*Moreover, for every matrix $A$ in $\mathrm{NZC}(\mathbb{Z}_+G[t])$, there is a matrix $A^\diamond$ over $\mathbb{Z}_+G$ such that $I-A$ is positive equivalent to $I-tA^\diamond$.*

*Proof.* The construction in [28, Sec. 7.2] produces from $A$ in $\mathrm{NZC}(\mathbb{Z}_+G[t])$ a matrix $A^\sharp$ over $\mathbb{Z}_+G$ such that there is a positive equivalence from $I-A$ to $I-tA^\sharp$. Then [28, Theorem 7.2] states (with different terminology) that $I-A$ and $I-B$ are positive equivalent if and only if $A^\sharp$ and $B^\sharp$ are SSE-$\mathbb{Z}_+G$. Now assume the Claim: for any square matrix $M$ over $\mathbb{Z}_+G$, $(tM)^\sharp$ is SSE-$\mathbb{Z}_+G$ to $M$. Then we have

$$\begin{aligned}
& A^\diamond \text{ and } B^\diamond \text{ are SSE over } \mathbb{Z}_+G \\
\iff\ & (tA^\diamond)^\sharp \text{ and } (tB^\diamond)^\sharp \text{ are SSE over } \mathbb{Z}_+G \\
\iff\ & I-tA^\diamond \text{ and } I-tB^\diamond \text{ are positive equivalent} \\
\iff\ & I-A \text{ and } I-B \text{ are positive equivalent.}
\end{aligned}$$

It suffices then to prove the Claim.

Suppose $M$ is square over $\mathbb{Z}_+G$. Let $\mathcal{G}$ be the $G$-labeled graph with adjacency matrix $M$. Let $\mathcal{H}$ be the $G$-labeled graph with adjacency matrix $C$ such that the vertices of $\mathcal{H}$ are the edges of $\mathcal{G}$, and $C$ is zero except that $C(a,b)$ is the label $g = g_a$ of edge $a$ in $\mathcal{G}$ if the terminal vertex of $a$ equals the inital vertex of $b$. By definition in [28, Sec.7] (note the "Special Case" remark above [28, (2.6)]), $(tM)^\sharp$ will be the adjacency matrix $C$ of $\mathcal{H}$. (The chosen ordering of indices to define an actual matrix won't affect the SSE-$\mathbb{Z}_+G$ class.) Explicitly, define matrices $R, S$, which are zero except for: $R(i,a) = 1$ if $i$ is the initial vertex of $a$; $S(a,j) = g_a$ if $j$ is the terminal vertex of the edge $a$. Then $M = RS$ and $C = SR$. $\square$

**Notational convention 3.11.** *For a matrix $A$ in $\mathrm{NZC}(\mathbb{Z}_+G[t])$, we will use $A^\diamond$ to denote a matrix over $\mathbb{Z}_+G$ such that $I - tA^\diamond$ is positive equivalent to $I - A$.*

The connection to shifts of finite type explained in [28] is less straightforward for $\mathrm{NZC}(\mathbb{Z}_+G[t])$ than for matrices over $t\mathbb{Z}_+G[t]$. However, $\mathrm{NZC}(R_+[t])$ is good for constructions (e.g., it is necessary for Proposition 3.16). Most importantly: if in the definition 3.10 of positive equivalence we replace $\mathrm{NZC}(R_+)$ with the set of square matrices over $t\mathbb{Z}_+G[t]$, then the implication (1) $\implies$ (2) of Theorem 3.2 would fail (see [28, Remark 6.4]).

The setting of positive equivalence has been useful for constructing conjugacies between SFTs and between $G$-SFTs [30, 31, 61, 62]. Positive equivalence constructions with matrices over $\mathbb{Z}_+G$ (not over $\mathbb{Z}_+G[t]$) are fundamental for the classification of $G$-SFTs up to equivariant flow equivalence in [32, 60].

Recall that a matrix is *nondegenerate* if it has no zero row and no zero column.

If row $i$ or column $i$ of a matrix is zero, then we say that the index $i$ is removable. For a square matrix $A$, let $A = A_0$. Given $A_k$, define $A_{k+1} = (0)$ if every index of $A_k$ is removable; otherwise, define $A_{k+1}$ to be the principal submatrix of $A_k$ on the nonremovable indices. For some $k$, $A_k = A_{k+1}$, and we call this matrix the *core* of $A$. A square matrix over $\mathbb{Z}_+G$ is always SSE over $\mathbb{Z}_+G$ to its core.

By Theorem 3.2, all matrices $A^\diamond$ over $\mathbb{Z}_+G$ with $I - tA^\diamond$ positive equivalent to a given $I - A$ lie in the same SSE-$\mathbb{Z}_+G$ class. So, given $A$, whether the core of $A^\diamond$ is $G$-primitive does not depend on the choice of $A^\diamond$. Similarly, given $A$, the following are equivalent: The choices are for $A^\diamond$, not for the core once $A^\diamond$ has been chosen.

1. Some choice of $A^0$ has core zero.
2. Every choice of $A^0$ has core zero.
3. Every $A^\diamond$ is SSE over $\mathbb{Z}_+G$ to $(0)$.
4. $I - A$ is positive equivalent to $I$.

**Some technical results.** The main purpose of this subsection is to prove its Propositions, which we need later in proofs.

Suppose $A$ is a square matrix over $t\mathbb{Z}_+G[t]$, say $A = \sum_{i=1}^{k} A_k t^k$, with the $A_k$

matrices over $\mathbb{Z}_+G$. As in [35], define the matrix

$$\mathcal{A}^{\square} = \begin{pmatrix} A_1 & A_2 & A_3 & \dots & A_{k-2} & A_{k-1} & A_k \\ I & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & I & 0 \end{pmatrix} . \tag{3.12}$$

*Remark* 3.13. If $B$ is a matrix with all entries in $\mathbb{Z}_+G[t]$, $\overline{B}$ is the matrix defined by applying the augmentation $\mathbb{Z}G \to \mathbb{Z}$ entrywise (Definition 3.42). Then for $A$ over $t\mathbb{Z}_+G[t]$, we have $\overline{(A^{\square})} = (\overline{A})^{\square}$, and the notation $\overline{\mathcal{A}^{\square}}$ is unambiguous.

**Lemma 3.14.** Suppose $A$ is a square matrix over $t\mathbb{Z}_+G[t]$. Then the matrices $I - A$ and $I - t\mathcal{A}^{\square}$ are positive equivalent.

*Proof.* The proof is clear from the case $k = 3$, as follows. The given multiplications by elementary matrices can be factored as a composition of basic positive equivalences.

$$\begin{pmatrix} I - tA_1 & -tA_2 & -tA_3 \\ -tI & I & 0 \\ 0 & -tI & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & tI & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ tI & I & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} I - A & -tA_2 - t^2A_3 & -tA_3 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

$$\begin{pmatrix} I & tA_2 + t^2A_3 & tA_3 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I - A & -tA_2 - t^2A_3 & -tA_3 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} I - A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

□

The next proposition is used in the proof of Lemma 3.27.

*Proposition* 3.15. Suppose $A$ is an $n \times n$ matrix in $\mathrm{NZC}(\mathbb{Z}_+G[t])$ and $d$ is the maximum degree of an entry of $A$. Then there is a matrix $A^\diamond$ over $\mathbb{Z}_+G$ such that the following hold.

1. $I - tA^\diamond$ is positive equivalent to $I - A$.

2. $A^\diamond$ is $m \times m$ with $m \leq nd$.

If $I - A$ is not positive equivalent to $I$, then in addition $A^\diamond$ can be chosen to be nondegenerate.

*Proof.* First suppose $A \in \mathrm{NZC}(\mathbb{Z}_+G[t])$. We claim $I - A$ is positive equivalent to a matrix over $t\mathbb{Z}_+G[t]$. This is stated for $\mathbb{Z}G = \mathbb{Z}$ in [28, Prop. 4.3], but the argument is for our purposes quite indirect, so we will sketch a proof. Suppose for a row $i$, the indices $j = j_1, \ldots, j_t$ are those such that $A(i,j)$ has nonzero constant term, $c_{i,j} \neq 0$. For $1 \leq s \leq t$, let $E_s$ be the $n \times n$ basic elementary matrix with $E(i, j_s) = c_{i,j_s}$. Then there is a positive equivalence from $I - A$ to $E_1E_2 \cdots E_t(I - A) := I - B_1$. $A$ and $B_1$ are equal outside row $i$. Now, if $M_i(A)$ denotes the maximum integer $k$ such that an entry of row $i$ of $A^k$ has nonzero constant term, then $M_i(B_1) \leq M_i(A) - 1$. Thus by iterating this process, we can produce an $n \times n$ matrix $B$ over $t\mathbb{Z}_+[t]$ such that $I - B$ is positive equivalent to $I - A$. Let $d_B$ be the maximum degree of an entry of $B$; then $d_B \leq d$.

Now by Lemma 3.14, the matrix $I - t\mathcal{B}^{\square}$ is positive equivalent to $I - B$ and hence to $I - A$, with size $nd_B \le nd$. Set $A^{\diamond} = \mathcal{B}^{\square}$. For the nondegeneracy condition, let $A^{\diamond}$ be the core of $\mathcal{B}^{\square}$. $\square$

The next proposition is used in the proof of Theorem 3.30.

*Proposition* 3.16. Suppose $I - A, I - B$ are matrices in $\mathrm{NZC}(\mathbb{Z}_+G[t])$ such that $A$ and $B$ are SSE over $\mathbb{Z}_+G[t]$. Suppose $A', B'$ are matrices over $\mathbb{Z}_+G$ such that $I - tA'$ and $I - tB'$ are positive equivalent respectively to $I - A$ and $I - B$. Then $A'$ and $B'$ are SSE over $\mathbb{Z}_+G$.

*Proof.* It suffices to prove the proposition in the case that there are matrices $R, S$ over $\mathbb{Z}_+G[t]$ such that $A = RS$ and $B = SR$. By Theorem 3.2, it suffices to show that $I - A$ is positive equivalent to $I - B$. To see this, using the "polynomial strong shift equivalence equations" of [28, Sec.4], we multiply by matrices below in the order given by subscripts. Each multiplication gives a positive equivalence.

$$\begin{pmatrix} I & 0 \\ S & I \end{pmatrix}_4 \begin{pmatrix} I & -R \\ 0 & I \end{pmatrix}_2 \begin{pmatrix} I - RS & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -S & I \end{pmatrix}_1 \begin{pmatrix} I & R \\ 0 & I \end{pmatrix}_3 = \begin{pmatrix} I & 0 \\ 0 & I - SR \end{pmatrix}$$

$$\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}_4 \begin{pmatrix} I & 0 \\ -B & I \end{pmatrix}_2 \begin{pmatrix} I & 0 \\ 0 & I - B \end{pmatrix} \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}_1 \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}_3 = \begin{pmatrix} I - B & 0 \\ 0 & I \end{pmatrix}$$

$\square$

**Infinite matrices.**

Let $R$ be a ring. $\mathrm{El}_n(R)$ is the group of $n \times n$ matrices which are products of basic elementary matrices over $R$. $\mathrm{GL}_n(R)$ is the group of $n \times n$ matrices invertible over $R$. For $R$ commutative, $\mathrm{SL}_n(R)$ is the subgroup of matrices in $\mathrm{GL}_n(R)$ with determinant 1. The group $\mathrm{GL}(R)$ is the direct limit group defined by the maps $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$, $U \mapsto U \oplus 1$. $\mathrm{El}(R)$ and (for $R$ commutative) $\mathrm{SL}(R)$ are the subgroups of $\mathrm{GL}(R)$ defined as direct limits of the groups $\mathrm{El}_n(R)$ and $\mathrm{SL}_n(R)$. We define finite square matrices $I - A, I - B$ to be $\mathrm{El}(R)$ equivalent if if there exist $j, k, n$ and matrices $U, V$ in $\mathrm{El}_n(R)$ such that $U(I - (A \oplus 0_j))V = I - (B \oplus 0_k)$. $\mathrm{GL}(R)$ equivalence and $\mathrm{SL}(R)$ equivalence are defined in the same way.

For a finite square matrix $M$, let $M_\infty$ denote the infinite matrix which has upper left corner $M$ and agrees with $I$ in all other entries. The elements of $\mathrm{GL}(R)$ are naturally identified with the matrices $U_\infty$ such that $U$ is invertible. Similarly for $\mathrm{SL}(R)$ and $\mathrm{El}(R)$.

An equivalence $U(I - A)V$ with $U$ and $V$ in $\mathrm{GL}_n(R)$ produces an equivalence $U_\infty(I - A)_\infty V_\infty$ by matrices $U, V$ in $\mathrm{GL}(R)$. Likewise for $\mathrm{El}(R)$ and $\mathrm{SL}(R)$. Basic elementary equivalence, ZNC and positive equivalence can be defined for these infinite matrices in the obvious way, such that finite square matrices $I - A$ and $I - B$ are positive equivalent if and only if $(I - A)_\infty$ and $(I - B)_\infty$ are positive equivalent.

**Algebraic invariants via polynomial matrices.**

In this subsection we look at the earlier algebraic invariants in terms of the polynomial matrix presentations.

**Definition 3.17.** For a ring $R$ we say square matrices $M, N$ are $\mathrm{El}(R)$ equivalent if there are positive integers $j, k, n$ and matrices $U, V$ in $\mathrm{El}_n(R)$ such that $U(M \oplus I_j)V = N \oplus I_k$. $\mathrm{GL}(R)$ equivalence and $\mathrm{SL}(R)$ equivalence are defined in the same way.

Theorem 3.3 is an easy corollary of the main result of [45].

**Theorem 3.3.** *[45, Corollary 6.6] Suppose $R$ is a ring. Suppose $I-A$ and $I-B$ are matrices over $R[t]$; $A', B'$ are square matrices over $R$; and $I-A$ and $I-B$ are respectively $\mathrm{El}(R[t])$ equivalent to $I-tA'$ and $I-tB'$. Then the following are equivalent.*

1. *$A'$ and $B'$ are SSE over $R$.*
2. *$I-tA$ and $I-tB$ are $\mathrm{El}(R[t])$ equivalent.*

If $A$ is $n \times n$ over the group ring $\mathbb{Z}G[t]$, then matrix multiplication defines $(I-A) : (\mathbb{Z}G[t])^n \to (\mathbb{Z}G[t])^n$ and thereby the $\mathbb{Z}G[t]$ module $\mathrm{cok}(I-A)$. (The isomorphism class of the module depends in general on whether one chooses multiplication of row vectors or column vectors.)

*Proposition* 3.18. [45, Theorem 5.1][2] Suppose $A$ and $B$ are square matrices over a ring $R$. Then the following are equivalent.

1. $A$ and $B$ are SE over $R$.
2. The $R[t]$ modules cokernel $\mathrm{cok}(I-tA)$ and $\mathrm{cok}(I-tB)$ are isomorphic.
3. $I-tA$ and $I-tB$ are $\mathrm{GL}(R[t])$ equivalent.

Lastly, we consider the algebraic invariants for the periodic data. Proposition 3.18 (via condition (3)) shows that $\det(I-tA)$ is invariant under SE-$R$ for any

[2]See [45] for attributions; especially, (2) $\iff$ (3) is due to Fitting [48].

commutative ring $R$ (e.g. $\mathbb{Z}G$ for $G$ abelian). For any ring $R$, $R[[t]]$ denotes the ring of formal power series with coefficients in $R$, and the *generalized characteristic polynomial* $\mathrm{ch}(A)$ [58,63–65]) of a square matrix $A$ over $R$ is the element of $K_1(R[[t]])$ containing $I - tA$. Motivation for and a characterization of $\mathrm{ch}(A)$ are in [58, 65]. If $R$ is commutative, then $\det(I - tA)$ is a complete invariant for $\mathrm{ch}(A)$.

Recall the definitions (3.5) and (3.6) for $\mathcal{T}_A$ and $\kappa\mathcal{T}_A$. Given a ring $R$, let $C$ denote the additive subgroup (not the ideal) of $R$ generated by the set $\{ab - ba : a \in R, b \in R\}$. Let $\gamma : R \to R/C$ denote the corresponding epimorphism of additive groups. Let $\mathcal{T}_A/C$ denote $\sum_{n=1}^{\infty} \gamma(\mathrm{tr} A^n) t^n$. Following Sheiham [58, p.19], for a square matrix $A$ over $R$ define $\chi : A \mapsto \mathcal{T}_A/C$ .

*Proposition* 3.19. Suppose $G$ is a group and $A, B$ are square matrices over $\mathbb{Z}G$. Then

$$\mathcal{T}_A/C = \mathcal{T}_B/C \iff \kappa\mathcal{T}_A = \kappa\mathcal{T}_B \ . \tag{3.20}$$

If $I - tA$ and $I - tB$ are $El(\mathbb{Z}G[t])$ equivalent, or even just $El(\mathbb{Z}G[[t]])$ equivalent, then $\kappa\mathcal{T}_A = \kappa\mathcal{T}_B$.

*Proof.* The proof of the first claim is straightforward. For the second claim, note that for any ring $R$, $\chi$ factors through $\mathrm{ch}(A)$, as pointed out by Sheiham [58, Remark 2.9]. If $I - tA$ and $I - tB$ are $\mathrm{El}(R[t])$ equivalent, then they are $\mathrm{El}(R[[t]])$ equivalent, so $\mathrm{ch}(A) = \mathrm{ch}(B)$. In the case $R = \mathbb{Z}G$, this means $\mathcal{T}_A/C = \mathcal{T}_B/C$. □

With Theorem 3.3, Proposition 3.19 gives an alternate proof that $\mathcal{T}_A = \mathcal{T}_B$ when $A$ and $B$ are SSE over $R$. For $G$ a nonabelian group, we do not know if $\kappa\mathcal{T}_A$ determines $\mathrm{ch}(A)$.

## 3.4 Parry's question and SE-$\mathbb{Z}G$

*Parry's Question* 3.21. Suppose $G$ is a finite abelian group, $(X, S)$ is a mixing SFT and $\zeta$ is a fixed dynamical zeta function. Must there be only finitely many topological conjugacy classes of $G$ extensions of $(X, S)$, with $\zeta_\tau$ constructed from a skewing function $\tau$ as in (3.1), such that $\zeta_\tau = \zeta$?

Slightly different versions of Parry's question were recorded in [27, Sec. 5.3], [66, Question 31.1] and [34, Sec. 4.4, p.331]. The version above is matched to our notation. The other versions are equivalent, except that the SFT $(X, S)$ might be assumed mixing or only irreducible. Because $(X, S)$ is fixed, a map $X \to X$ implementing an isomorphism of $(X, S, \tau_1)$ and $(X, S, \tau_2)$ would have to be an automorphism of $(X, S)$, as in the language of [34, Sec. 4.4]. (For work on a related problem, in which the skewing function $f$ is Hölder into the real numbers, see [67].)

We will address the following version of Question 3.21.

*Question* 3.22. Suppose $G$ is a nontrivial finite group and $A$ is a $G$-primitive matrix over $\mathbb{Z}_+G$. Let $\mathfrak{M}(A)$ be the collection of $G$-primitive matrices $B$ over $\mathbb{Z}_+G$ such that

1. the matrices $\overline{A}$ and $\overline{B}$ are SSE over $\mathbb{Z}_+$, and
2. the matrices $B$ and $A$ have the same periodic data, $P_B = P_A$ as in (3.7) (if $G$ is abelian, this means $\det(I - tB) = \det(I - tA)$).

Must $\mathfrak{M}(A)$ contain only finitely many SSE-$\mathbb{Z}_+G$ classes?

In Question 3.22, the condition that $A$ be $G$-primitive adds the requirement that the extension be a mixing extension – the central case. A negative answer to (3.22) gives a negative answer to (3.21). The condition that $\overline{A}$ and $\overline{B}$ are SSE over $\mathbb{Z}_+$ captures up to isomorphism the extensions of Question 3.21 (we can recode them to this form) and also includes every $(X', S', \tau')$ such that $(X', S')$ is topologically conjugate to $(X, S)$ and $\tau'$ gives the correct periodic data. This does not change the set of isomorphism classes of extensions, because isomorphism classes of $G$-extensions of $(X', S')$ pull back bijectively under topological conjugacy to isomorphism classes of $G$-extensions of $(X, S)$. Also, we have broadened Parry's question to include nonabelian groups. We add the condition that $G$ be nontrivial for linguistic simplicity. If $G$ is trivial, then the answer to (3.21) is trivially 'yes', so we no longer need to exclude this case when giving a negative answer. If $G$ is nontrivial and $A$ is $G$-primitive, then the extension must have positive entropy, and there is nothing more to say about excluding a case of finitely many orbits.

Parry[3] with an unpublished example showed that nonisomorphic skew products over a mixing SFT could share the same zeta function $\zeta_\tau$. His question followed the study of dozens of examples, and grew out a study of cocycles describing how Markov measures change under a flow equivalence of SFTs as in [68].

A natural way to attack Question 3.22 is to consider how the algebraic relations SE-$\mathbb{Z}G$ and SSE-$\mathbb{Z}G$ can refine a prescribed $\det(I - tA)$. If the refinement is infinite, then there is an issue of constructing $G$-primitive matrices realizing an infinite class

[3] Descriptions of Parry's work and motivation are based on a review of email correspondence 2002-2006 between Boyle and Parry.

on which the algebraic invariants differ. In Section 3.5, we'll carry out this program at the level of SSE-$\mathbb{Z}G$, when $\mathrm{NK}_1(\mathbb{Z}G)$ is not trivial. In this case, for *every* $A$ the answer is negative.

If $\mathrm{NK}_1(\mathbb{Z}G)$ is trivial, then SE-$\mathbb{Z}G$ and SSE-$\mathbb{Z}G$ are equivalent, by Theorem 4.5. By appeal to SE-$\mathbb{Z}G$ invariants, Theorem 3.4 below gives a negative answer to Parry's question for every $G$, regardless of whether $\mathrm{NK}_1(\mathbb{Z}G)$ is trivial. However, in contrast to the SSE-$\mathbb{Z}G$ invariants, the SE-$\mathbb{Z}G$ invariants do not provide an infinite refinement of the periodic data of $A$ for *every* $A$. We will give examples for which the data $\det(I - tA)$ determines the SE-$\mathbb{Z}G$ class of $A$.

$\mathfrak{M}(A)$ in the statement of Theorem 3.4 was defined in Question 3.22.

**Theorem 3.4.** *Suppose $G$ is a nontrivial finite group. There is a $G$-primitive matrix $A$ over $\mathbb{Z}G$ such that $\mathfrak{M}(A)$ contains infinitely many SE-$\mathbb{Z}G$ equivalence classes.*

*Proof.* We will define some matrices over $\mathbb{Z}G[t]$. Let $u = \sum_{g \in G} g \in \mathbb{Z}G$. Fix $g$ an element of $G$ distinct from the identity $e$. Set $s = ut \in \mathbb{Z}G[t]$ and $w = et$. Below, $p_k$ in $\mathbb{Z}G[t]$ will depend on $k \in \mathbb{Z}_+$, with $p_0 = 0$. Given $r$, $E_{ij}(r)$ denotes the basic elementary matrix of appropriate size which equals $r$ in the $i, j$ entry and otherwise equals $I$. Define $5 \times 5$ matrices equal to $I$ except that $U(3,4) = U(3,5) = 1 = V(5,1) = V(4,1)$. $U$ will act by adding column 3 to columns 4 and 5. $V$ will act by adding row 1 to rows 4 and 5. Define

$$C_k = \begin{pmatrix} 4s & s & s & 0 & 0 \\ 4s & s & s & 0 & 0 \\ 4s & 2s & 2s & 0 & 0 \\ 0 & 0 & 0 & w & p_k \\ 0 & 0 & 0 & 0 & w \end{pmatrix}$$

$$D_k = U^{-1}C_kU = \begin{pmatrix} 4s & s & s & s & s \\ 4s & s & s & s & s \\ 4s & 2s & 2s & 2s-w & 2s-w-p_k \\ 0 & 0 & 0 & w & p_k \\ 0 & 0 & 0 & 0 & w \end{pmatrix}$$

$$F_k = VD_kV^{-1} = \begin{pmatrix} 2s & s & s & s & s \\ 2s & s & s & s & s \\ 2w+p_k & 2s & 2s & 2s-w & 2s-w-p_k \\ 2s-w-p_k & s & s & s+w & s+p_k \\ 2s-w & s & s & s & s+w \end{pmatrix}.$$

We will choose $p_k$ to be a sum of $k$ monomials, $p_k = (e-g)(t^{n_1} + \cdots + t^{n_k})$. Define $A = F_0$. Then $A = t\mathcal{A}^\square$ and $\mathcal{A}^\square$ is $G$-primitive. For each $k$, we have $\overline{p_k} = 0$, and therefore $\overline{F_k} = A$. We will arrange the following.

1. The $\mathbb{Z}G[t]$ modules $\mathrm{cok}(I - C_k)$ are pairwise not isomorphic.

2. For each $k$, there is a matrix $B_k$ over $t\mathbb{Z}_+G[t]$ and a finite string of matrices $F_k = B_{(0)}, B_{(1)}, \ldots, B_{(m)} = B_k$ such that the following hold.

(a) $B_k^\square$ is $G$-primitive .

(b) For $1 \le i \le m$, $I - B_{(i)}$ equals $E_i(I - B_{(i-1)})$ or $(I - B_{(i-1)})E_i$, for some basic elementary matrix $E_i$ with offdiagonal entry in $t\mathbb{Z}G[t]$.

(c) For $0 \le i \le m$, $\overline{B_{(i)}}$ has all entries in $t\mathbb{Z}_+[t]$.

Suppose we have these conditions. For each $k$, the $\mathbb{Z}G[t]$ modules $\mathrm{cok}(I - C_k)$, $\mathrm{cok}(I - F_k)$ and, by 2(b), $\mathrm{cok}(I - B_k)$ are isomorphic. Therefore the $\mathbb{Z}G[t]$ modules $\mathrm{cok}(I-B_k)$, are, by (1), pairwise not isomorphic. Therefore the $G$-primitive matrices $B_k^\square$ are pairwise not shift equivalent over $\mathbb{Z}G$. However, the elementary equivalences of 2(b) over $\mathbb{Z}G[t]$ push down to elementary equivalences over $\mathbb{Z}[t]$, and by 2(c) these are positive equialences over $\mathbb{Z}[t]$. Therefore each $\overline{B_k}$ is SSE over $\mathbb{Z}_+[t]$ to $A$, and the first condition in the Question 3.22 definiton of $\mathfrak{M}(A)$ is satisfied. For the second condition, note by 2(b) and Proposition 3.19 that for each $k$ the matrices $B_k^\square$ and $F_k^\square$ have the same periodic data. $F_k^\square$ and $C_k^\square$ also have the same periodic data. By the block structure of $C_k$, the entry $p_k$ has no effect on the traces of powers of $C_k$. Thus every $C_k$ has the periodic data of $C_0$, which is that of $A$. This shows the second condition in the Question 3.22 definition of $\mathfrak{M}(A)$ is satisfied. So, it remains to arrange the conditions (1) and (2) above.

For condition (2), consider the multiplication of $I - F_k$ from the right by matrices $E_{25}(s), E_{25}(s^2), \dots, E_{25}(s^k)$, producing say a matrix $I - G_k$. These push down to a positive equivalence from $I - \overline{F_k}$ to $I - \overline{G_k}$. We have

$$G_k(3,5) = (2s - w - p_k) + 2s^2 + 2s^3 + \cdots + 2s^{k+1}$$

$$G_k(4,5) = (s + p_k) + s^2 + s^3 + \cdots + s^{k+1} \ .$$

Thus for suitable $p_k$ of the specified form, these two entries of $G_k$ will lie in $\mathbb{Z}_+G[t]$. Apply the same procedure with $E_{21}$ in place of $E_{25}$ to likewise address the sign issue for the 1,3 and 1,4 entries. The resulting matrix is our $B_k$.

Finally, we address condition (1). For $h$ in $G$, let $\widetilde{h}$ be the $|G| \times |G|$ permutation matrix which is the image of $G$ under the left regular representation. This induces a map $M \mapsto \widetilde{M}$ sending $5 \times 5$ matrices over $\mathbb{Z}G[t]$ to $5|G| \times 5|G|$ matrices over $\mathbb{Z}[t]$. Suppose there is an isomorphism of $\mathbb{Z}G[t]$ modules $\mathrm{cok}(I - C_k) \to \mathrm{cok}(I - C_j)$. Let the homomorphism $\mathbb{Z}[t] \to \mathbb{Z}$ induced by $t \mapsto 1$ send a matrix $I - \widetilde{C}$ to $I - C'$. Then there is an induced isomorphism of $\mathbb{Z}$ modules (abelian groups), $\mathrm{cok}(I - C'_j) \to \mathrm{cok}(I - C'_k)$. The lower right $2|G| \times 2|G|$ block of $I - C'_k$ has the block form $\left(\begin{smallmatrix} 0 & k(I-P) \\ 0 & 0 \end{smallmatrix}\right)$, where $P$ is $\widetilde{g}$. From the block diagonal form of $C_j$ and $C_k$ we conclude that $\mathrm{cok}(k(I - P))$ and $\mathrm{cok}(j(I - P))$ are isomorphic groups.

But, let $m$ be the order of $g$ in $G$ and let $c = |G|/m$. $P$ is conjugate by a permutation matrix to the direct sum of $c$ copies of a matrix $C$, where $C$ is an $m \times m$ cyclic permutation matrix. $I_m - C$ is $\mathrm{SL}_m\mathbb{Z}$-equivalent to $I_{m-1} \oplus 0_1$. Therefore $\mathrm{cok}(k(I - P))$ is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^{(m-1)c} \oplus \mathbb{Z}^c$, and for positive integers $j \neq k$, $\mathrm{cok}(j(I - P))$ and $\mathrm{cok}(k(I - P))$ cannot be isomorphic. This contradiction finishes the proof. $\square$

**Lemma 3.23.** Suppose $G$ is a finite group, and let $u = \sum_g g$. Suppose $A$ and $B$ are matrices over $\mathbb{Z}G$ with some powers $A^p, B^q$ all of whose entries lie in $u\mathbb{Z}$. Suppose that $\overline{A}$ and $\overline{B}$ are SE over $\mathbb{Z}$. Then $A$ and $B$ are SE over $\mathbb{Z}G$.

*Proof.* For any matrix $M$ over $\mathbb{Z}G$, we have $uM = u\overline{M}$. So, $A^p = u(1/|G|)\overline{A^p} = u(1/|G|)\overline{A}^p$, with $(1/|G|)\overline{A}^p$ having integer entries. For $k > 0$, $A^{p+k} = u(1/|G|)\overline{A}^{p+k}$. Without loss of generality, we suppose $p = q$. Suppose $R, S$ gives an SE over $\mathbb{Z}$ of $\overline{A}^\ell$ and $\overline{B}^\ell$:

$$\overline{A}^\ell = RS\ ,\ \overline{B}^\ell = SR\ ,\ \overline{A}R = R\overline{B}\ ,\ S\overline{A} = \overline{B}S\ .$$

Define $\widetilde{R} = A^pR = u(1/|G|)\overline{A}^pR$ and $\widetilde{S} = B^pS = u(1/|G|)\overline{B}^pS$. Then

$$\begin{aligned}
\widetilde{R}\widetilde{S} &= \Big(u(\frac{1}{|G|}\overline{A}^pR)\Big)\Big(u(\frac{1}{|G|}\overline{B}^pS)\Big) = u(\frac{1}{|G|}\overline{A}^pR\overline{B}^pS) \\
&= u(\frac{1}{|G|}\overline{A}^pRS\overline{A}^p) = u(\frac{1}{|G|}\overline{A}^{2p+\ell}) = A^{2p+\ell}\ , \quad \text{and} \\
A\widetilde{R} &= A\Big(u(\frac{1}{|G|}\overline{A}^pR)\Big) = u\frac{1}{|G|}\overline{A}^{p+1}R \\
&= u\frac{1}{|G|}\overline{A}^pR\overline{B} = u\frac{1}{|G|}\overline{A}^pRB = \widetilde{R}B
\end{aligned}$$

(for the last line, note that $u$ lies in the center of $\mathbb{Z}G$). Likewise, $\widetilde{S}\widetilde{R} = B^{2p+\ell}$ and $B\widetilde{S} = \widetilde{S}A$ . $\square$

It is easy to construct matrices $A$ over $\mathbb{Z}_+G$ such that some power $A^p$ has all entries in $u\mathbb{Z}G$. For example, take $A$ over $u\mathbb{Z}G$; or let $A = B + N$ where $B$ is over $u\mathbb{Z}_+G$ and $N$ over $\mathbb{Z}G$ is nilpotent with $uN = 0$. If $B$ here is also $G$-primitive and $B - N$ has all entries over $\mathbb{Z}_+G$, then $A$ will be $G$-primitive.

**Lemma 3.24.** Suppose $A$ is $n \times n$ over $\mathbb{Z}G$, with $m = |G|$. Let $\tau_k$ denote $\mathrm{tr}(A^k)$, with $\tau_{k,g}$ the integers such that $\tau_k = \sum_{g\in G} \tau_{k,g}g$ . Then the following are equivalent.

1. There is $p$ in $\mathbb{N}$ such that $A^p$ has all entries in $u\mathbb{Z}G$.

2. $m\tau_{k,e} = \overline{\tau_k}$, for $1 \le k \le mn$,

Now suppose a positive power of $A$ has all entries in $u\mathbb{Z}G$ and $B$ is a matrix over $\mathbb{Z}G$ such that (i) $B$ and $A$ have the same periodic data or (ii) $B$ is SE over $\mathbb{Z}G$ to $A$. Then some positive power of $B$ has all entries in $u\mathbb{Z}G$. Consequently, for $\mathcal{R} = \mathbb{Z}$ or $\mathcal{R} = \mathbb{Z}_+$: if $\overline{A}$ and $\overline{B}$ are SE-$\mathcal{R}$, then $A$ and $B$ are SE-$\mathcal{R}G$.

*Proof.* We use $\widetilde{A}\colon \mathbb{Z}^{mn} \to \mathbb{Z}^{mn}$ constructed as in Appendix 3.8. Let $W$ be the subspace of $\mathbb{Z}^{mn}$ corresponding to $(u\mathbb{Z}G)^n$. $A$ has a positive power with all entries in $u\mathbb{Z}G$ if and only if $\widetilde{A}$ has a power which maps $\mathbb{Z}^{mn}$ into $W$ if and only if $\widetilde{A}$ restricted to the complementary invariant subspace is nilpotent. This holds if and only if the sequences $(\mathrm{tr}(\widetilde{A}^k))_{1\le k\le mn}$ and $(\mathrm{tr}((\widetilde{A}|_W)^k))_{1\le k\le mn}$ are equal. We have $\mathrm{tr}(\widetilde{A}^k) = m\tau_{k,e}$ and (because $A$ acts on $(u\mathbb{Z}G^n)$ exactly as $\overline{A}$ acts on $\mathbb{Z}^n$) $\mathrm{tr}((\widetilde{A}|_W)^k) = \overline{\tau_k}$. This proves the equivalence of (1) and (2).

Then (i) holds because (1) $\iff$ (2) shows (1) depends only on the periodic data. Although the periodic data need not be an invariant of SE-$\mathbb{Z}G$ when $G$ is nonabelian, if matrices $A, B$ are SE-$\mathbb{Z}G$ then for every large enough $\ell \in \mathbb{N}$ there are $R, S$ over $\mathbb{Z}G$ such that $A^\ell = RS$ and $B^\ell = SR$, and then $A^{2\ell} = (A^\ell R)S$ and $B^{2\ell} = S(A^\ell R)$. Clearly if $A^\ell$ is over $u\mathbb{Z}G$, then so is $B^{2\ell}$. The final claim follows now from Lemma 3.23. $\square$

*Proposition* 3.25. Suppose $G$ is a finite abelian group. Set $u = \sum_g g$. Let $\mathcal{Z}_u$ denote the set of polynomials of the form $1 + \sum_{i=1}^k c_i u t^i$, with each $c_i$ in $\mathbb{Z}$. Suppose $A$ and

$B$ are square matrices over $\mathbb{Z}G$ such that $\det(I-tA)$ and $\det(I-tB)$ lie in $\mathcal{Z}_u$, and $\overline{A}$ and $\overline{B}$ are SE over $\mathbb{Z}$. Then $A$ and $B$ are SE over $\mathbb{Z}G$.

*Proof.* By the Cayley-Hamilton Theorem, for all large $n$ the matrices $A^n$ and $B^n$ have entries in $u\mathbb{Z}$. The theorem then follows from Lemma 3.23. □

Proposition 3.25 applies to any $A$ all of whose entries are integer multiples of $u$; for example, $A = (e+g)$ with $G = \{e, g\} = \mathbb{Z}/2\mathbb{Z}$.

In the case of $A$ satisfying the assumptions of Proposition 3.25, with $\mathrm{NK}_1(\mathbb{Z}G)$ trivial, to answer Parry's question we are left with the open problem: for $A$ $G$-primitive, can the refinement the SSE-$\mathbb{Z}G$ class of $A$ by SSE-$\mathbb{Z}G_+$ be infinite? In the case $G = \{e\}$ ($\mathbb{Z}G = \mathbb{Z}$), that question remains open more than 40 years after Williams' original paper [1].

*Remark* 3.26. In [4, Theorem 7.1], Parry proved that for $G$ compact and $X$ an irreducible SFT if $f, g : X \to G$ are Hölder with equal weights on all periodic points, then $f$ and $g$ are Hölder cohomologous. If one assumes only that the $f$ and $g$ weights are conjugate, Parry shows then the existence of an isometric automorphism $\phi$ of $G$ such that $\phi f$ and $g$ are cohomologous [4, Theorem 6.5]. Parry also gives an example with $G$ finite of two cocycles having conjugate weights for which the isomorphism is necessary, although the example is not mixing [4, Section 10].

We note now that in general $\phi$ cannot be chosen to be the identity even if the extension is mixing (i.e., presented by a $G$-primitive matrix $A$ over $\mathbb{Z}_+G$). For example, let $G$ be a finite group having an outer automorphism $\varphi$ for which $\varphi$ preserves all conjugacy classes of $G$. Such groups exist (see [69]); for example, the group $LP(1, \mathbb{Z}/8)$ consisting of all linear permutations $x \mapsto \sigma x + \tau$ on $\mathbb{Z}/8$, with $\sigma, \tau$ in $\mathbb{Z}/8$, is such a group. Let $A$ be primitive over $\mathbb{Z}_+G$, and $\tau$ denote the corresponding edge labeling on the graph of $\overline{A}$ coming from $A$. Then $\phi\tau$ is another edge labeling, and $\phi\tau$ and $\tau$ have conjugate weights on all periodic points. However, $\phi\tau$ and $\tau$ are not cohomologous. If they were, then because they are defined by edge labelings of an irreducible graph, by [4, Lemma 9.1] there would be a function $\gamma : X_A \to G$ such that $\gamma(x)$ depends only on the initial vertex of $x$ and $\phi\tau = \gamma^{-1}\tau\gamma$ . Let $\nu$ be a vertex and let $g$ in $G$ be such that $g = \gamma(x)$ when $x_0$ has initial vertex $\nu$. Now for every word $x_0 \dots x_k$ beginning and ending at $\nu$: if $h = \tau(x_0) \cdots \tau(x_k)$, then $\phi(h) = g^{-1}hg$. Because $A$ is $G$-primitive, every element of $G$ occurs as such an $h$, and therefore $\phi$ is an inner automorphism. This contradiction shows $\phi\tau$ and $\tau$ are not cohomologous.

## 3.5 Parry's question and SSE-$\mathbb{Z}G$

In this section, we prove the following result, which gives a strong negative answer to Parry's question (3.21) whenever $\mathrm{NK}_1(\mathbb{Z}G) \neq 0$. (See Appendix 3.9 for a description of the finite $G$ with nontrivial $\mathrm{NK}_1(\mathbb{Z}G)$.)

**Theorem 3.5.** *Let $G$ be a finite group such that $\mathrm{NK}_1(\mathbb{Z}G) \neq 0$. Let $(X, \sigma)$ be a mixing shift of finite type and let $\tau : X \to G$ be a continuous function defining a mixing $G$-extension $(X_\tau, \sigma_\tau)$ of $(X, T)$.*

*Then there is an infinite family of $G$-extensions of $(X, T)$ which are eventually conjugate as $G$-extensions to $(X_\tau, \sigma_\tau)$ and which are pairwise not isomorphic $G$-*

*extensions. If $G$ is abelian, then they all have the same dynamical zeta function.*

Theorem 3.5 will be proved as a corollary to the following result.

**Theorem 3.6.** *Suppose $G$ is a finite group and $A$ is a $G$-primitive matrix with spectral radius $\lambda > 1$ and $\mathrm{NK}_1(\mathbb{Z}G) \neq 0$. Let $A$ be a $G$-primitive matrix.*

*Then there is an infinite family $\{A_i : i \in \mathbb{N}\}$ of $G$-primitive matrices which are pairwise not SSE over $\mathbb{Z}G$ but such that for all $i$ the following hold:*

1. *$A_i$ is SE over $\mathbb{Z}_+G$ to $A$.*

2. *$\overline{A_i}$ is SSE over $\mathbb{Z}_+$ to $\overline{A}$.*

3. *If $G$ is abelian, then $\det(I - tA_i) = \det(I - tA)$ .*

To prove Theorem 3.6, we first will work to establish a rather technical result, Proposition 3.30. Below, we will use the notations of (3.43) and the definitions (3.12), (3.45) and (3.46) of a matrix $\mathcal{A}^{\square}$, a $G$-primitive matrix and the spectral radius $\lambda_A$ of a square matrix over $\mathbb{Z}G$ or $\mathbb{Z}G[t]$. For a polynomial $p$ over $\mathbb{Z}G$, $\lambda_p$ is the spectral radius of the $1 \times 1$ matrix $(p)$. For a polynomial matrix $M = M(t)$, we let $M(1)$ denote its evaluation at $t = 1$.

**Lemma 3.27.** Suppose $n > 1$ and $A$ is an $n \times n$ matrix over $t\mathbb{Z}_+G[t]$ with spectral radius $\lambda > 1$ and with $\mathcal{A}^{\square}$ $G$-primitive . Given $\epsilon > 0$, there exists a positive integer $m_0$ such that for any $d \geq m_0$ there is an $n \times n$ matrix $C$ over $t\mathbb{Z}_+G[t]$ such that $I - C$ is positive equivalent to $I - tA$ and

$$c_{11kg} > (\lambda - \epsilon)^k \ , \quad \text{for } m_0 \leq k \leq d \ , \quad \text{for all } g \in G \ .$$

*Proof.* We will produce $C$ in three stages.

STAGE 1. Because $A(1)$ is $G$-primitive, by [32, Lemma 6.6] there is a positive equivalence with respect to the ordered ring $(\mathbb{Z}G, \mathbb{Z}_+G)$ from $I - A(1)$ to a matrix $I - H$ such that $H$ is a matrix over $\mathbb{Z}_+G$ with no zero entry. Lift this positive equivalence with respect to $(\mathbb{Z}G, \mathbb{Z}_+G)$ to a positive equivalence with respect to $(\mathbb{Z}G[t], \mathbb{Z}_+G[t])$ from $I - tA$ to a matrix $I - L$, with $L$ a matrix over $t\mathbb{Z}_+G[t]$ with every entry nonzero.

STAGE 2. In this stage, given $\epsilon > 0$ we produce an $n \times n$ matrix $B$ over $t\mathbb{Z}_+G[t]$ with no zero entry such that $I - tA$ is $\mathbb{Z}G[t]$ positive equivalent to $I - B$ and the $B(n, n)$ entry has spectral radius greater than $\lambda - \epsilon/2$.

For this, we define $n \times n$ matrices $B_1, B_2, \ldots$ recursively. We set $B_1$ to be the matrix $L$ produced in Stage 1. In block form, let $B_1 = \left(\begin{smallmatrix} M & u \\ v & f \end{smallmatrix}\right)$, in which $f$ is $1 \times 1$. A matrix $B_k$ will have a block form

$$B_k = \begin{pmatrix} M & u \\ v^{(k)} & f^{(k)} \end{pmatrix} . \tag{3.28}$$

Given $B_k$, define $B_{k+1}$ by the equivalence

$$I - B_{k+1} = \begin{pmatrix} I & 0 \\ v^{(k)} & I \end{pmatrix} \begin{pmatrix} I - M & -u \\ -v^{(k)} & 1 - f^{(k)} \end{pmatrix} = \begin{pmatrix} I - M & -u \\ -v^{(k)}M & 1 - f^{(k)} - v^{(k)}u \end{pmatrix} .$$

This defines a positive equivalence from $I - B_k$ to $I - B_{k+1}$. By induction, for all $k$, $B_k$ is in $\mathcal{M}$ and has no zero entry; $B_k^{\square}$ is $G$-primitive ; and $B_{k+1}$ is a matrix over $t\mathbb{Z}_+G[t]$ with block form

$$B_{k+1} = \begin{pmatrix} M & u \\ vM^k & f + v(I + M + \cdots + M^{k-1})u \end{pmatrix} .$$

Because $A$ is $G$-primitive , by the condition (3) in Theorem 3.7 we have a positive real number $c$ such that $\mathrm{tr}(A^j) > c\lambda^j(g_1 + \cdots + g_m)$ for all large $j$. Because $M^\square$ is a proper principal submatrix of the $G$-primitive matrix $(B_1)^\square$, which has spectral radius $\lambda$, we have $\lambda_M < \lambda$. Choose $\delta > 0$ such that $\delta < \epsilon/2$ and $\lambda_M < \lambda - \delta$. For all large $j$,

$$\mathrm{tr}(M^j) < (\lambda - \delta)^j(g_1 + \cdots + g_m) \ .$$

Because $M^k$ has entries in $t^k\mathbb{Z}G[t]$ and $u$ has entries in $t\mathbb{Z}G[t]$, if $j \leq k$ then

$$\mathrm{tr}\big((A^\square)^j\big) \ = \ \mathrm{tr}\big((M^\square)^j\big) + \mathrm{tr}\big((f^{(k)})^\square)^j\big) \ .$$

It follows that for all large $k$, for $j \in \{k-1, k\}$,

$$\mathrm{tr}\big(((f^{(k)})^\square)^j\big) \geq (\lambda - \delta)^j(g_1 + \cdots + g_m) \ . \qquad (3.29)$$

Consequently, $f^{(k)}$ is $G$-primitive for all large $k$.

Let $\lambda^{(k)}$ be the spectral radius of $(f^{(k)})$. Let $d$ be the maximum degree of an entry of $B_1$. From the block form (3.28) we see that $f^{(k)}$ has degree at most $dk$. Then by Proposition 3.15, we can use a version of $p^\square$ which is a $dk \times dk$ matrix $Q$ over $\mathbb{Z}_+G$. Then for $q = dk/m$, the matrix $\overline{Q}$ is $q \times q$ over $\mathbb{Z}_+$ with spectral radius $\lambda_Q = \lambda^{(k)}$. Using (3.29), we have

$$\lambda^{(k)} = \lambda_Q \geq \Big(\frac{1}{q}\mathrm{tr}(Q^k)\Big)^{1/k} \geq \Big(\frac{m}{dk}(\lambda - \delta)^k m)\Big)^{1/k} \ .$$

Because $0 < \delta < \epsilon/2$, It follows that $\lambda^{(k)} > \lambda - \epsilon/2$ for all large $k$.

STAGE 3. We define $n \times n$ matrices $P_1, P_2, \ldots$ over $t\mathbb{Z}_+[t]$ recursively. The recursive step is the same as in Stage 2, but with row 1 in Stage 3 playing the role of row $n$ in Stage 2. In block form, we write $P_1 = \left(\begin{smallmatrix} s & w \\ x & Q \end{smallmatrix}\right)$, with $s$ being $1 \times 1$. We take $P_1 = B$ from Stage 2 and set $q = P_1(n,n)$. The $1 \times 1$ matrix $\begin{pmatrix} q \end{pmatrix}$ has $q^{\square}$ $G$-primitive with spectral radius $\lambda_q$ such that $0 < \lambda - \lambda_q < \epsilon/2$ .

A matrix $P_k$ will have a block form

$$P_k \;=\; \begin{pmatrix} s^{(k)} & w^{(k)} \\ x & Q \end{pmatrix}$$

and given $P_k$ we define $P_{k+1}$ by

$$I - P_{k+1} \;=\; \begin{pmatrix} 1 & w^{(k)} \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 - s^{(k)} & -w^{(k)} \\ -x & I - Q \end{pmatrix} = \begin{pmatrix} 1 - s^{(k)} - w^{(k)}x & -w^{(k)}Q \\ -x & I - Q \end{pmatrix} .$$

By induction,

$$P_{k+1} \;=\; \begin{pmatrix} s + w(I + Q + \cdots + Q^{k-1})x & wQ^k \\ x & Q \end{pmatrix}$$

and $q = P_{k+1}(n,n)$. As in Proposition 3.48 , let $(\tau_j)$ be the sequence from $\mathbb{Z}_+G$ such that

$$\sum_{k=1}^{\infty} q^k = \sum_{j=1}^{\infty} \tau_j t^j \ .$$

Appealing to Proposition 3.48, choose positive $c', d'$ such that $\tau_j > c'(\lambda_q)^j(g_1 + \cdots + g_m)$ for all $j \geq d'$. Pick $g, h$ in $G$ and positive integers $n_1, n_2$ satisfying $gt^{n_1} \leq P_1(1,n)$

and $ht^{n_2} \leq P_1(n,1)$. Then for $k > d'$,

$$\begin{aligned} P_{k+1}(1,1) &\geq P_1(1,n)\ (Q(n,n))^{k-1}\ P_1(n,1) \\ &\geq gt^{n_1} \left( \sum_{j=d'}^{k-1} c'(\lambda_q)^j (g_1 + \cdots + g_m) t^j \right) ht^{n_2} \\ &\geq \sum_{j=d'+n_1+n_2}^{k+n_1+n_2-1} \left( \frac{c'}{(\lambda_q)^{n_1+n_2}} \right) (\lambda_q)^j (g_1 + \cdots + g_m) t^j \\ &> \sum_{j=d'+n_1+n_2}^{k+n_1+n_2-1} \left( \frac{c'}{(\lambda_q)^{n_1+n_2}} \right) \left( \lambda - \frac{\epsilon}{2} \right)^j (g_1 + \cdots + g_m) t^j\ . \end{aligned}$$

Let $m_0$ be the smallest $j$ such that $j \geq d' + n_1 + n_2$ and

$$\left( \frac{c'}{(\lambda_q)^{n_1+n_2}} \right) \left( \lambda - \frac{\epsilon}{2} \right)^j \ > \ (\lambda - \epsilon)^j\ .$$

Then given $d \geq m_0$, for $k = d$ we have $P_k(1,1) > \sum_{j=m_0}^{d} (\lambda - \epsilon)^j (g_1 + \cdots + g_m) t^j$ .

This finishes the proof of the lemma. $\square$

*Proposition* 3.30. Suppose $A$ is an $n \times n$ $G$-primitive matrix over $\mathbb{Z}_+ G$, $n > 1$ and $1 < \beta < \lambda_A$. Then there is a positive integer $r_0$ such that the following holds. If $r \geq r_0$ and $I - Q$ is a matrix in $\mathrm{GL}(k, \mathbb{Z}[t])$ such that

(i) $|q_{ijsg}| \leq \beta^s$ for all $i, j, s, g$, and

(ii) $Q \in \mathcal{M}(t^r \mathbb{Z}[t])$

then the matrix $\begin{pmatrix} I - Q & 0 \\ 0 & I - tA \end{pmatrix}$ is $\mathrm{El}(\mathbb{Z}G[t])$ equivalent to an $(m+k) \times (m+k)$ matrix $I - B$ over $\mathbb{Z}G[t]$ such that

1. $B$ has entries in $t\mathbb{Z}_+ G[t]$

2. $B^{\square}$ is $G$-primitive

3. if $\overline{Q} = 0$, then $\overline{B}^{\square}$ is SSE over $\mathbb{Z}_+$ to $\overline{A}$.

*Proof.* We use $\sim$ to denote $\mathrm{El}(\mathbb{Z}G[t])$ equivalence. First, note that if $I-F$ is a matrix over $\mathbb{Z}G[t]$ with block form $I - F = \begin{pmatrix} I-Q & -X \\ 0 & I-C \end{pmatrix}$ such that $I - C \sim I - tA$, then the invertibility of $I-Q$ implies $I - F \sim \begin{pmatrix} I-Q & 0 \\ 0 & I-tA \end{pmatrix} = I - (Q \oplus tA)$, since

$$\begin{pmatrix} I-Q & -X & 0 \\ 0 & I-C & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} (I-Q)^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I-Q) \end{pmatrix} \begin{pmatrix} I & X & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I-Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I-Q)^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} I-Q & 0 & 0 \\ 0 & I-C & 0 \\ 0 & 0 & I \end{pmatrix} .$$

Next, given $\beta$, let $\epsilon = (\lambda_A - \beta)/2$ and let $m_o$ be the integer of the conclusion of Lemma 3.27 given $A$ and $\epsilon$. Suppose $I - Q \in \mathrm{GL}(k, \mathbb{Z}[t])$ and $Q$ satisfies (i) and (ii). Pick $r \geq m_0$ such that for all $s \geq r$, $(\lambda_A - \epsilon)^s > 2k\lceil \beta^s \rceil + 1$ . Let $d$ be an integer such that $d > r$ and $d \geq \mathrm{degree}(Q)$. Now take $I - C$ from Lemma 3.27, positive equivalent to $I - tA$, such that

$$c_{11gm} \geq 2k\lceil \beta^m \rceil + 1 \ , \quad \text{for } r \leq m \leq d \ \text{ and for all } g \ .$$

Let $u = \sum_g g$. Let $\alpha = \sum_{m=r}^{d} \lceil \beta^m \rceil u t^m$ . Consider a matrix in block form,

$$H = \begin{pmatrix} Q & X \\ 0 & C \end{pmatrix} = \left(\begin{array}{cccc|cccc} q_{11} & q_{12} & \cdots & q_{1k} & \alpha & 0 & \cdots & 0 \\ q_{21} & q_{22} & \cdots & q_{2k} & \alpha & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ q_{k1} & q_{k2} & \cdots & q_{kk} & \alpha & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & 0 & \cdots & 0 & c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & c_{n1} & c_{n2} & \cdots & c_{nn} \end{array}\right).$$

Define a matrix $V$ with matching block structure, $V = \begin{pmatrix} I_k & 0 \\ Y & I_n \end{pmatrix}$, in which the top row of $Y$ has every entry 1 and the other entries of $Y$ are zero, and in which $I_j$ as usual denotes a $j \times j$ identity matrix. Define $B = V^{-1}HV$. We have

$$B = \left(\begin{array}{cccc|cccc} q_{11}+\alpha & q_{12}+\alpha & \cdots & q_{1k}+\alpha & \alpha & 0 & \cdots & 0 \\ q_{21}+\alpha & q_{22}+\alpha & \cdots & q_{2k}+\alpha & \alpha & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ q_{k1}+\alpha & q_{k2}+\alpha & \cdots & q_{kk}+\alpha & \alpha & 0 & \cdots & 0 \\ \hline x-\eta_1 & x-\eta_2 & \cdots & x-\eta_k & x & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{21} & \cdots & c_{21} & c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ c_{n1} & c_{n1} & \cdots & c_{n1} & c_{n1} & c_{n2} & \cdots & c_{nn} \end{array}\right) \tag{3.31}$$

in which $x = c_{11} - k\alpha$ and $\eta_j = q_{1j} + q_{2j} + \cdots + q_{kj}$. Then $x \geq (k+1)u(t^d + \cdots + t^r)$,

$x - \eta_j \geq u(t^d + \cdots + t^r)$ and $q_{ij} + \alpha \geq 0$. Because $x$ is $G$-primitive and $C$ is $G$-primitive, it follows easily that $B$ is $G$-primitive. Also, since $I - B = V^{-1}(I - H)V$, the matrix $I - B$ is $\mathrm{El}(\mathbb{Z}G[t])$ equivalent to $I - H$, and therefore to $I - t(Q \oplus C)$.

Finally, suppose $\overline{Q} = 0$. We must show $\overline{B^{\square}}$ is SSE over $\mathbb{Z}_+$ to $A$. Clearly $A$ and $\overline{C^{\square}}$ are SSE over $\mathbb{Z}_+$. The matrices $B$ and $C$ have all entries in $t\mathbb{Z}_+[t]$. Thus by Remark 3.13, $\overline{B^{\square}} = \overline{B}^{\square}$ and $\overline{C^{\square}} = \overline{C}^{\square}$. Therefore it suffices to show that $\overline{B}^{\square}$ and $\overline{C}^{\square}$ are SSE over $\mathbb{Z}_+$. By Proposition 3.16, this will follow if we show $\overline{B}$ is SSE over $\mathbb{Z}_+[t]$ to $\overline{C}$.

Because $\overline{Q} = 0$, we have $\overline{B} = \overline{H'}$, where $H'$ is the matrix obtained from $H$ by replacing the entries $q_{ij}$ and $\eta_j$ in the display (3.31) with zero. Let $D$ be the lower right hand block of the $2 \times 2$ block matrix $B$. $H'$ is SSE over $\mathbb{Z}_+G[t]$ to $C$, since

$$C = \begin{pmatrix} Y & I_n \end{pmatrix} \begin{pmatrix} X \\ D \end{pmatrix} \quad \text{and} \quad H' = \begin{pmatrix} X \\ D \end{pmatrix} \begin{pmatrix} Y & I_n \end{pmatrix} .$$

Therefore $\overline{B} = \overline{H'}$ is SSE over $\mathbb{Z}_+[t]$ to $\overline{C}$. This finishes the proof. □

**Lemma 3.32.** Suppose $G$ is a finite group, $N$ is nilpotent $n \times n$ over $\mathbb{Z}G$ and $r \in \mathbb{N}$. Then there is a matrix $M_r$ over $t^r\mathbb{Z}G[t]$ such that $\overline{M_r} = 0$ and $I - M_r$ is $\mathrm{El}(n, \mathbb{Z}G[t])$-equivalent to $I - t^r N$. Given $N$, the matrices $M_r$ can be chosen such that the coefficients of all entries are bounded above independent of $r$.

*Proof.* Suppose $N$ is $n \times n$. Because $\overline{N}$ is nilpotent over $\mathbb{Z}$, we can take $U$ in $\mathrm{SL}_n(\mathbb{Z}) = \mathrm{El}_n(\mathbb{Z})$ such that the matrix $N_1 = U^{-1}\overline{N}U$ is upper triangular with zero diagonal. Given $r$, for $1 \leq i < n$, let $W$ be $n \times n$ with $W(i,j) = -t^r N_1(i,j)$ if $i < j$ and $W = I$ otherwise. Set $W = W_1 W_2 \cdots W_{n-1}$; then $W \in \mathrm{El}(n, \mathbb{Z}G[t])$

and $\overline{W(I - t^r N_1)} = I$. Let $M_r$ be the matrix over $t\mathbb{Z}G[t]$ such that $I - M_r = WU^{-1}(I - t^r N)U = W(I - t^r N_1)$. Then $I - \overline{M}_r = \overline{I - M_r} = \overline{W(I - t^r N_1)} = I$, so $\overline{M}_r = 0$. The boundedness claim is clear from the construction. □

**Lemma 3.33.** Suppose $G$ is a finite group and $A$ is a $G$-primitive matrix with spectral radius $\lambda > 1$ and $N$ is nilpotent over $\mathbb{Z}G$. Then for all sufficently large $r$ in $\mathbb{N}$, the matrix $\begin{pmatrix} I - tA & 0 \\ 0 & I - t^r N \end{pmatrix}$ is $\mathrm{El}(\mathbb{Z}G[t])$-equivalent to a matrix $I - B$ such that $B$ has entries in $t\mathbb{Z}G_+[t]$ and $B^\square$ is $G$-primitive and $\overline{B^\square}$ is SSE over $\mathbb{Z}_+$ to $\overline{A}$.

*Proof.* Pick $\beta$ such that $1 < \beta < \lambda$. Let $r_0$ be the integer of Proposition 3.30, which depends on $A$ and $\beta$. Let $\{M_r\}$ be the uniformly bounded family given for $\{t^r N\}$ by Lemma 3.32. Then for all large $r \in \mathbb{N}$, $r \geq r_0$ and the matrix $Q = M_r$ satisfies $|q_{ijs}| \leq \beta^s$ for all $i, j, g, s$. Because $t^r N$ is nilpotent, the matrix $I - t^r N$ is invertible over $\mathbb{Z}G[t]$. Now Lemma 3.33 follows from Proposition 3.30. □

Given $r \in \mathbb{N}$, define $V_r : NK_1(\mathbb{Z}G) \to NK_1(\mathbb{Z}G)$ by $V_r : [I - tN] \mapsto [I - t^r N]$, and $F_r : NK_1(\mathbb{Z}G) \to NK_1(\mathbb{Z}G)$ by $F_r : [I - tN] \mapsto [I - tN^r]$. The map $V_r$ is often called the Verschiebung operator, and $F_r$ the Frobenius operator.

**Lemma 3.34.** Let $G$ be a finite group and $r \in \mathbb{N}$ be such that $r$ and $|G|$ are relatively prime. Then the map $V_r : NK_1(\mathbb{Z}G) \to NK_1(\mathbb{Z}G)$ is injective.

*Proof.* One may check directly that $F_r V_r(x) = rx$ for all $x \in NK_1(\mathbb{Z}G)$. By a result of Weibel [70, 6.5, p. 490], the order of every element in $NK_1(\mathbb{Z}G)$ must be a power of $|G|$. Thus the map $F_r V_r$ is injective for $r$ relatively prime to $|G|$, and $V_r$ is as well. □

*Proof of Theorem 3.6.* Because $\mathrm{NK}_1(\mathbb{Z}G)$ is nontrivial, it is infinite [20]. Given $j \in \mathbb{N}$, let $N_1, \dots, N_j$ be nilpotent over $\mathbb{Z}G$ with the matrices $I - tN_j$ representing distinct classes of $\mathrm{NK}_1(R)$. For a sufficiently large such $r$, Lemma 3.33 applies to each $t^r N_i$, giving $B_i$ satisfying the conclusions of the lemma. We take $r$ which in addition is relatively prime to $|G|$; then the matrices $I - t^r N_i$ will represent distinct classes of $\mathrm{NK}_1(\mathbb{Z}G)$, by Lemma 3.34. Let $A_i = B_i^{\square}$. Condition (2) holds as part of Lemma 3.33. Condition (1) holds because (i) adding a nilpotent direct summand to a matrix does not affect its SE class and (ii) for $G$-primitive matrices, SE over $\mathbb{Z}G$ is equivalent to SE over $\mathbb{Z}_+G$ (Prop. 3.51).

By Theorem 4.5, the matrices $A_i$ are pairwise not SSE over $\mathbb{Z}G$. Condition (3) holds because $\det(I - tA_i) = \det(I - tA)\det(I - tN_i)$ and $\det(I - tN_i)$ here must be 1 by Prop. 3.52. $\square$

*Proof of Theorem 3.5.* Let $A$ be a $G$-primitive matrix defining a $G$ extension which is isomorphic to that defined by $\tau$ and let $A_i$ be the $G$-primitive matrices provided by Theorem 3.6. By condition (1) of Theorem 3.6 and Proposition 3.50, these $G$ extensions of $(X, T)$ are all eventually conjugate to $(X_\tau, \sigma_\tau)$. By condition (2), the $A_i$ define $G$ extensions which are conjugate to $G$-extensions defined from $(X, T)$. Because the $A_i$ are not SSE over $\mathbb{Z}G$, they cannot be SSE over $\mathbb{Z}_+G$, so their extensions (and hence their conjugate extensions from $(X, T)$) are pairwise not isomorphic. Lastly, they satisfy condition (3), which for abelian $G$ is a well defined invariant of SSE over $\mathbb{Z}G$ (and even SE over $\mathbb{Z}G$) and therefore is carried over to the isomorphic versions defined over $(X, T)$.

□

## 3.6 Open problems

*Realization Problems* 3.35. This set of problems for the algebraic analysis of mixing finite group extensions of SFTs involves understanding the range of the algebraic invariants.

1. Suppose $G$ is finite group, $A$ is $G$-primitive and $N$ is a nilpotent matrix over $\mathbb{Z}G$. Must $A \oplus N$ be SSE over $\mathbb{Z}G$ to a $G$-primitive matrix?

   (The methods for Section 3.5 and [8, Radius Theorem] might be useful. The answer to the corresponding problem for matrices over subrings of $\mathbb{R}$ is positive [35].)

2. Given a finite abelian group $G$, characterize the polynomials $\det(I-tA)$ arising from $G$-primitive matrices $A$ over $\mathbb{Z}G$.

   (For $\mathbb{Z}G = \mathbb{Z}\{e\} = \mathbb{Z}$, this is solved [9].)

3. Given a finite group $G$, characterize the trace series $\mathcal{T}_A$ and conjugate trace series $\kappa\mathcal{T}_A$ arising from $G$-primitive matrices $A$ over $\mathbb{Z}G$.

4. Let $G$ be a finite abelian group. Suppose $A$ is a $G$-primitive matrix over $\mathbb{Z}_+G$, and $B$ is a matrix over $\mathbb{Z}G$ such that $\det(I-tA) = \det(I-tB)$. Must $B$ be shift equivalent over $\mathbb{Z}G$ to a $G$-primitive matrix?

   (For analogues involving $\mathbb{R}$ and $\mathbb{Z}$, see [35].)

5. Let $G$ be a finite group. Suppose $A$ is a $G$-primitive matrix over $\mathbb{Z}_+G$, and $B$

is a matrix over $\mathbb{Z}G$ with the same conjugate trace series (3.6), $\kappa\mathcal{T}_A = \kappa\mathcal{T}_B$. Must $B$ be shift equivalent over $\mathbb{Z}G$ to a $G$-primitive matrix?

*Algebraic Study* 3.36. For square matrices $A$ over $\mathbb{Z}G$, $G$ a finite group, make a satisfactory algebraic study of the $\mathbb{Z}G[t]$-modules $\operatorname{cok}(I-tA)$ and the associated $\mathbb{Z}G$-modules $\operatorname{cok}(I-A)$. (The latter arise as invariants of $G$-equivariant flow equivalence [32].)

*Sufficiency of invariants* 3.37. The following questions are open even for $G = \{e\}$.

1. For $G$-primitive matrices, what invariants must be added to SSE-$\mathbb{Z}G$ to imply SSE-$\mathbb{Z}_+G$?

2. Prove or disprove: for $G$ nontrivial, every SSE-$\mathbb{Z}G$ class of $G$-primitive matrices contains infinitely many SSE-$\mathbb{Z}_+G$ classes.

## 3.7 $G$-SFTs defined from matrices: left vs. right action

In this section we describe how $G$ extensions of SFTs are defined from matrices over $\mathbb{Z}_+G$, and the corresponding classifying role of strong shift equivalence of the matrices over $\mathbb{Z}_+G$ (SSE-$\mathbb{Z}_+G$). In the process, we correct (see the Erratum 3.39 below) an error in the corresponding definition in [32]. Given $X \times G$, the map $g : (x, h) \mapsto (x, hg)$ defines a right action of $G$ on $X \times G$, and the map $g : (x, h) \mapsto (x, gh)$ defines a left action of $G$ on $X \times G$.

There are corresponding notations for presenting a $G$ extension. Suppose $T : X \to X$ is a homeomorphism and $\tau : X \to G$ is continuous. For the left action on $X \times G$ we define the group extension $T_{\ell,\tau} : X \times G \to X \times G$ by $T_\ell : (x,h) \mapsto (T(x), h\tau(x))$. For the right action we define $T_{r,\tau} : X \times G \to X \times G$ by $T_r : (x,h) \mapsto (T(x), \tau(x)h)$. Each commutes with its associated $G$ action.

In the case of the left $G$ action, continuous functions $\tau, \tau'$ from $X \times G$ to $G$ are *cohomologous* if there is a continuous $\gamma : X \to G$ such that for all $x$, $\tau'(x) = \gamma^{-1}(x)\tau(x)\gamma(Tx)$. In the case of the right action, the cohomology equation is $\tau'(x) = \gamma(Tx)\tau(x)\gamma^{-1}(x)$

Now suppose $A$ is square over $\mathbb{Z}_+G$. The matrix $\overline{A}$ over $\mathbb{Z}_+$ is defined from $A$ by applying the augmentation map $\sum_g n_g g \mapsto \sum_g n_g$ entrywise. We view $\overline{A}$ as the adjacency matrix of a directed graph. If the set of edges from vertex $i$ to vertex $j$ is nonempty, label them by elements of $G$ to match $A(i,j) = \sum_g n_g g$: for each $g$, exactly $n_g$ edges are labeled $g$. Let $\tau_A : X_{\overline{A}} \to G$ be the continuous function which sends $x = \dots x_{-1}x_0x_1\dots$ to the label of the edge $x_0$, denoted $\ell(x_0)$. We use $T_{\ell,A}$ and $T_{r,A}$ to denote $T_{\ell,\tau}$ and $T_{r,\tau}$ with $\tau = \tau_A$.

In the case of the left $G$ action, with $T$ the shift on $X_A$, for the corresponding $G$ extension $T_{\ell,A}$ defined on $X_{\overline{A}} \times G$, for $n > 0$ we have

$$\begin{aligned} T_\ell^n : (x,h) &\mapsto (T^n x, h\tau_A(x)\cdots\tau_A(T^{n-1}x)) \\ &= (T^n x, h\ell(x_0)\cdots\ell(x_{n-1})) \ . \end{aligned}$$

Here a weight $w = \ell(x_0)\ell(x_1)\cdots\ell(x_{n-1})$ is the product of the labels along the edge-path $x_0x_1\cdots x_{n-1}$. If $A^n(i,j) = \sum_g n_g g$, then the number of edge paths with initial vertex $i$, terminal vertex $j$ and weight $g$ is equal to $n_g$. This is the connection of matrix and group extension behind the following result of Parry (see [32, Prop.

2.7.1]). In the statement, $\tau_A \sim \tau_B \circ \varphi$ means there is a continuous $\gamma : X_{\overline{A}} \to G$ such that $\tau_B(\varphi(x)) = \gamma^{-1}(x)\tau_A(x)\gamma(\sigma_A x)$. In the proposition we need only assume that $G$ is a discrete group, not necessarily finite. In this case, any continuous function into $G$ will then be locally constant.

*Proposition* 3.38. Let $G$ be a discrete group. The following are equivalent for matrices $A$ and $B$ over $\mathbb{Z}_+G$.

1. $A$ and $B$ are SSE over $\mathbb{Z}_+G$.

2. There is a homeomorphism $\varphi\colon X_{\overline{A}} \to X_{\overline{B}}$ such that $\varphi\sigma_{\overline{A}} = \sigma_{\overline{B}}\varphi$ and $\tau_A \sim \tau_B \circ \varphi$.

3. The $G$-SFTs $T_{\ell,A}$ and $T_{\ell,B}$ are $G$-conjugate.

Explanation for all this is in [32]– after correction of the following error.

*Erratum* 3.39. In [32, Sec. 2.4], the group extensions (skew products) were defined as extensions for the right $G$ action on $X \times G$. They should instead be extensions for the left $G$ action on $X \times G$. Consequently two other changes should be made.

1. In paragraph 2 of [32, Sec. 2.7], "draw an edge from $(g, i)$ to $(\ell(e)g, j)$" should be "draw an edge from $(g, i)$ to $(g\ell(e), j)$".

2. In the final sentence of paragraph 2 of [32, Sec. 2.7], "$(h, j) \mapsto (hg, j)$" should be "$(h, j) \mapsto (gh, j)$" .

*Remark* 3.40. We record below some relations among matrices and extensions. We use $A'$ to denote the transpose of a matrix $A$; if $A$ has entries in $\mathbb{Z}_+G$, we let $A^{\mathrm{opp}} =$

$A^{\mathrm{o}}$ be the matrix defined by applying entrywise the map $\sum_g n_g g \mapsto \sum_g n_g g^{-1}$. (This map is an isomorphism from $\mathbb{Z}G$ to its opposite ring.)

1. $(T_{\ell,A})^{-1}$ and $T_{\ell,(A')^{\mathrm{o}}}$ are conjugate $G$ extensions.

2. The $G$ extension $T_{r,A}$ is conjugated to the $G$ extension $T_{\ell,A^{\mathrm{o}}}$, by the map $(x,h) \mapsto (x,h^{-1})$. (Note, $(x,hg) \mapsto (x,(hg)^{-1}) = (x, g^{-1}h^{-1})$ .)

3. $T_{r,A}$ and $T_{r,B}$ are conjugate $G$ extensions $\iff$ $A^{\mathrm{o}}$ and $B^{\mathrm{o}}$ are SSE-$\mathbb{Z}_+G$.

4. $A$ and $B$ SSE-$\mathbb{Z}_+G$ $\implies$ $(A')^{\mathrm{o}}$ and $(B')^{\mathrm{o}}$ are SSE-$\mathbb{Z}_+G$.

   (Note: $A = RS, B = SR \implies (A')^{\mathrm{o}} = (S')^{\mathrm{o}}(R')^{\mathrm{o}}, (B')^{\mathrm{o}} = (R')^{\mathrm{o}}(S')^{\mathrm{o}}$.)

5. For $G$ nonabelian, for $A$ and $B$ SSE-$\mathbb{Z}_+G$:

   $A'$ and $B'$ need not be SSE-$\mathbb{Z}_+G$; $A^{\mathrm{o}}$ and $B^{\mathrm{o}}$ need not be SSE-$\mathbb{Z}_+G$.

   (See Example 3.41).

6. For $G$ nonabelian, for $T_{\ell,A}$ and $T_{\ell,B}$ conjugate $G$-extensions:

   $T_{r,A}$ and $T_{r,B}$ need not be conjugate $G$-extensions.

*Example* 3.41. Let $A \sim B$ mean $A$ and $B$ are SSE-$\mathbb{Z}_+G$. We give an example here of $A \sim B$ with $A^{\mathrm{opp}} \not\sim B^{\mathrm{opp}}$ and $A' \not\sim B'$. We use $G$ the group of permutations on $\{1,2,3,4\}$, in which $gh$ is defined by $(gh)(x) = g(h(x))$. Let $M[x,y,z]$ denote a matrix $M$ with $M(1,2) = x, M(2,3) = y, M(3,1) = z$ and $M = 0$ otherwise. In $G$, define $a = (143), b = (123), c = (12)(34), d = (13)(24)$; then $abc = e$ and $a^{-1}b^{-1}c^{-1} = d \neq e$. Set $A = M[a,b,c]$ and $B = M[e,e,e] = B^{\mathrm{opp}}$. Then $A \sim M[e,e,abc] = B$, but $A^{\mathrm{opp}} = M[a^{-1},b^{-1},c^{-1}] \sim M[e,e,a^{-1}b^{-1}c^{-1}] = M[e,e,d]$, and $M[e,e,d] \not\sim B$ (e.g. by Proposition 3.44). Therefore $A^{\text{opp}} \not\sim B^{\text{opp}}$. Similarly, $B' \sim B$, and $A' \sim M[e,e,cab] = M[e,e,d] \not\sim B$.

## 3.8 $G$-primitive matrices and shift equivalence

**Primitivity for matrices over $\mathbb{Z}G$.**

In this section, $G$ is a finite group. We will spell out some basic facts around the regular representation of $G$, our use of the Perron Theorem and SE over $\mathbb{Z}_+G$.

Let $m = |G|$. Fix an enumeration of the elements of $G$, $G = \{g_1, \dots, g_m\}$, with $g_1 = e$, the identity element. If $x = \sum_i n_i g_i \in \mathbb{Z}G$, then its image under the augmentation map is $\overline{x} = \sum_i n_i$.

**Definition 3.42.** For vectors $v$ and matrices $M$ over $\mathbb{Z}G[t]$ (perhaps over just $\mathbb{Z}G$), we define $\overline{v}$ and $\overline{M}$ by applying the augmentation map entrywise, $\sum_{s=0}^{S} \sum_g m_{sg} g t^s \mapsto \sum_{s=0}^{S} \sum_g m_{sg} t^s$ .

**Notational convention 3.43.** *Given a matrix $A$ over $\mathbb{Z}G$, define $a_{ij} = A(i,j)$ and $a_{ijk} = A^k(i,j)$, and let $a_{ijkg}$ be the integers such that*

$$A^k(i,j) = a_{ijk} = \sum_g a_{ijkg}\, g \ .$$

*Define $\overline{a}_{ij} = \sum_g a_{ij1g}$, i.e., $\overline{A}(i,j) = \overline{a}_{ij}$. The uppercase - lowercase correspondence above producing a given $A$ may be used for other letters as well.*

Let $e_i$ denote the size $m$ column vector whose $i$th entry is 1 and whose other entries are zero. Define an isomorphism of additive groups $p : \mathbb{Z}G \to \mathbb{Z}^m$ by the rule $\sum_i n_i g_i \mapsto \sum_i n_i e_i$. We carry over the usual partial order on $\mathbb{Z}^m$: for $x = \sum_i n_i g_i$

we say $x \geq 0$ if $n_i \geq 0$ for all $i$, and we write $x \gg 0$ if $n_i > 0$ for all $i$. When we use an order relation for vectors or matrices, we mean that it holds entrywise. For example, $x \gg 0$ in $\mathbb{Z}G$ if and only if $p(x) > 0$ in $\mathbb{Z}^m$. We also carry over the usual notion of convergence in $\mathbb{Z}^m$: a sequence of elements $x^{(k)} = \sum_i n_i^{(k)} g_i$ converges to $x = \sum_i n_i g_i$ iff $\lim_k n_i^{(k)} = n_i$ for each $i$. Convergence of vectors or matrices over $\mathbb{Z}G$ is by definition entrywise convergence.

For $1 \leq r \leq m$, define $m \times m$ permutation matrices $P_r, Q_r$ by the rules

$$\begin{aligned} P_r(i,j) &= 1 \quad \text{iff} \quad g_r g_j = g_i \\ Q_r(i,j) &= 1 \quad \text{iff} \quad g_j g_r = g_i \ . \end{aligned}$$

Then $P_r(p(g_j)) = p(g_r g_j)$ and $Q_r(p(g_j)) = p(g_j g_r)$. The map $g_r \mapsto P_r$ is the regular representation of $G$ given by its action on itself by multiplication from the left; similarly for $Q_r$ and right multiplication. For $x = \sum_j n_j g_j \in \mathbb{Z}G$, we similarly define $\rho(x)$ to be the $m \times m$ matrix over $\mathbb{Z}$ which presents multiplication by $x$ from the left. That is, the following diagram commutes, with $\rho(x) = \sum_j n_j P_j$ :

$$\begin{array}{ccccccc} \mathbb{Z}G & \xrightarrow{x} & \mathbb{Z}G & \qquad & y & \longrightarrow & xy \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} & & \big\downarrow & & \big\downarrow \\ \mathbb{Z}^m & \xrightarrow{\rho(x)} & \mathbb{Z}^m & & p(y) & \longrightarrow & \rho(x)p(y) \end{array}$$

Column 1 of the matrix $\rho(x)$ is $p(x) = \begin{pmatrix} n_1 \\ \vdots \\ n_m \end{pmatrix}$, since $p(x) = p(xe) = \rho(x)p(e) = \rho(x)e_1$. For each $j$, column $j$ of $\rho(x)$ is $Q_j \begin{pmatrix} n_1 \\ \vdots \\ n_m \end{pmatrix}$, since column $j$ of $\rho(x)$ equals

$$\rho(x)e_j = \rho(x)p(g_j) = p(xg_j) = Q_j p(x) \ .$$

Now suppose $A$ is $\ell \times n$ over $\mathbb{Z}G$. Define an $\ell m \times nm$ matrix $\widetilde{A}$, with a block form of $m \times m$ blocks, in which the $ij$ block is $\rho(a_{ij})$. If $A, B$ over $\mathbb{Z}G$ have compatible sizes for matrix multiplication, then $\widetilde{A}\widetilde{B} = \widetilde{AB}$. Letting $\kappa$ be defined as in Definition 3.2, we pause to record some facts used in Section 3.2 to discuss the periodic data (3.7).

*Proposition* 3.44. Let $G$ be a finite group, with $m = |G|$. Suppose $A$ is an $n \times n$ matrix over $\mathbb{Z}G$. Let $\eta(t) \in \mathbb{Z}[t]$ be the characteristic polynomial of $\widetilde{A}$. Then $\eta(A) = 0$, and

1. the finite sequence $(\mathrm{tr}(A^k))_{1 \le k \le mn}$, determines $(\mathrm{tr}(A^k))_{1 \le k < \infty}$ .
2. the finite sequence $(\kappa(\mathrm{tr}(A^k)))_{1 \le k \le mn}$ determines $(\kappa(\mathrm{tr}(A^k)))_{1 \le k < \infty}$.

If $A$ and $B$ are matrices SSE over $\mathbb{Z}G$, then $(\kappa(\mathrm{tr}(A^k))_{1 \le k < \infty} = (\kappa(\mathrm{tr}(B^k))_{1 \le k < \infty}$.

*Proof.* $\eta(A) = 0$ because $A \mapsto \widetilde{A}$ defines an embedding of the ring of $n \times n$ matrices over $\mathbb{Z}G$ into the ring of $nm \times nm$ matrices over $\mathbb{Z}$. The coefficients of $\eta$ are determined by the finite sequence $(\mathrm{tr}(\widetilde{A}^k))_{1 \le k \le mn}$, which equals $(m \sum_i a_{iike})_{1 \le k \le mn}$, which is determined by $(\kappa(\mathrm{tr}(A^k))_{1 \le k \le mn}$. The claims (1,2) then follow because $\eta(A) = 0$ gives integers $c_1, \dots, c_{nm}$ such that $\mathrm{tr}(A^k) = c_1 \mathrm{tr}(A^{k-1}) + \cdots + c_{nm} \mathrm{tr}(A^{k-nm})$ for all $k > mn$. It suffices to prove the final claim in the case $A = RS, B = SR$ for matrices $R, S$ over $\mathbb{Z}G$. Then

$$\mathrm{tr}(RS) = \sum_{i,j,g,h} r_{ij1g} s_{ji1h} gh \quad \text{and} \quad \mathrm{tr}(SR) = \sum_{i,j,g,h} s_{ji1h} r_{ij1g} hg \ .$$

Because $gh = h^{-1}(hg)h$, it follows that $\kappa(\mathrm{tr}(RS)) = \kappa(\mathrm{tr}(SR))$ . For $k > 1$, we have $A^k = (A^{k-1}R)S$ and $B^k = S(A^{k-1}R)$ . The conclusion follows. □

**Definition 3.45.** For a matrix $A$ over $\mathbb{R}G$ (e.g., $A$ over $\mathbb{Z}G$), we say $A$ is $G$-primitive if $A$ is square, $A \ge 0$ and, for some $k > 0$, $A^k \gg 0$.

Clearly $A$ is $G$-primitive if and only if $\widetilde{A}$ is primitive, since $A \gg 0$ is equivalent to $\widetilde{A} > 0$. (For an example, consider $G = \mathbb{Z}/2$, $g \neq e$ and $A = (5g)$, giving $\widetilde{A} = \left(\begin{smallmatrix} 0 & 5 \\ 5 & 0 \end{smallmatrix}\right)$; here $\overline{A}$ is primitive but $A$ is not $G$-primitive .) The spectral radius of a real matrix $M$ is denoted $\lambda_M$. The matrices $\overline{A}$ and $\widetilde{A}$ have the same spectral radius.

**Definition 3.46.** Let $G$ be a finite group. The spectral radius $\lambda_A$ of a square matrix $A$ over $\mathbb{Z}G$ is defined to be $\lambda_{\overline{A}} = \lambda_{\widetilde{A}}$. The spectral radius $\lambda_A$ of a square matrix $A$ over $\mathbb{Z}G[t]$ is defined to be the spectral radius of $A^\square$.

Naturally, for $A$ square over $\mathbb{Z}G$, we have $\lambda_A = \overline{\lim}_k \max_{i,j,g} |a_{ijkg}|^{1/k}$ .

**Theorem 3.7.** *Suppose $G$ is a finite group, $G = \{g_1, \ldots, g_m\}$ with $g_1 = e$, the identity element of $G$. Suppose $A$ is an $n \times n$ matrix over $\mathbb{Z}_+G$ such that its augmentation $\overline{A}$ is irreducible. Let $\lambda = \lambda_A$. For $i$ in $\{1, \ldots, n\}$, set $H_i = \cup_k \{g \in G : a_{iikg} > 0\}$. Then the following statements are true.*

1. *The sets $H_i$ are conjugate subgroups of $G$.*

2. *The following are equivalent.*

   (a) *$\widetilde{A}$ is primitive.*

   (b) *$A$ is $G$-primitive.*

   (c) *Let $\overline{\ell}, \overline{r}$ denote positive left and right eigenvectors of $\overline{A}$ such that $\overline{\ell}\overline{r} = (1)$ (these vectors exist because $\overline{A}$ is irreducible). Then*

   $$\lim_k \left(\frac{1}{\lambda} A\right)^k \;=\; (g_1 + \cdots + g_m)\frac{1}{m}\overline{r}\overline{\ell} \ .$$

   (d) *With the notation $\operatorname{tr}(A^k) = \sum_g \tau_{kg} g$, the following conditions hold:*

*i. There are relatively prime $j, k$ such that $\tau_{ke} > 0$ and $\tau_{je} > 0$.*

*ii. There exists $i$ such that $H_i = G$.*

3. *If $G$ is abelian and $\overline{A}$ is irreducible, then the polynomial* $\det(I - tA)$ *determines whether $A$ is $G$-primitive .*

*Remark* 3.47. It follows from the Perron theorem that the convergence in (3) above is exponentially fast.

*Proof of Theorem 3.7.* (1) Given $i$, there exists a diagonal matrix $D$, with each diagonal entry an element of $G$, such that $D^{-1}AD$ has all entries in $H_i$ [32, Proposition 4.4]. As in [32], it follows easily that the $H_i$ are conjugate subgroups of $G$.

(2) (a) $\iff$ (b) This was part of the paragraph before the theorem.

(b) $\implies$ (c) Let $u$ denote $g_1 + \cdots + g_m$. The augmentation matrix $\overline{A}$ is primitive, because $A$ is $G$-primitive. Therefore $((1/\lambda)\overline{A})^k$ converges to $\overline{r}\overline{\ell}$. Define size $n$ vectors over $\mathbb{R}_+G$ by setting $\ell = u\overline{\ell}$ and $r = u\overline{r}$. If $x = \sum_i n_i g_i \in \mathbb{Z}G$, then $xu = (\sum_i n_i)u = ux$. Therefore

$$Ar = Au\overline{r} = u\overline{A}\overline{r} = u\lambda\overline{r} = \lambda r$$

and likewise $\ell A = u\lambda\overline{\ell} = \lambda\ell$ . These eigenvectors lift to eigenvectors $\widetilde{\ell}, \widetilde{r}$ of $\widetilde{A}$. Explicitly, $\widetilde{\ell} = (\widetilde{\ell}_1, \dots, \widetilde{\ell}_n)$ in which $\widetilde{\ell}_j$ is the size $m$ row vector $p(u\overline{\ell_j})$; every entry of $\widetilde{\ell}_j$ equals $\overline{\ell}_j$. Likewise, every entry of $\widetilde{r}_j$ equals $\overline{r}_j$. We have $(\widetilde{\ell}\widetilde{r}) = m(\overline{\ell}\overline{r})$. Only now do we appeal to the primitivity of $\widetilde{A}$, which guarantees

$$\lim_k \left(\frac{1}{\lambda}\widetilde{A}\right)^k = \frac{1}{m}\left(\widetilde{r}\widetilde{\ell}\right) .$$

Translated back to $A$, this becomes

$$\lim_k \Big(\frac{1}{\lambda}A\Big)^k \;=\; (g_1 + \cdots + g_m)\Big(\frac{1}{m}\overline{r}\overline{\ell}\Big) \;.$$

(c) $\implies$ (d) Obvious.

(d) $\implies$ (b) The subgroups $H_i$ are conjugate, so (d) implies that $H_i = G$ for every $i$. Now suppose $j, k$ are relatively prime with $\tau_{ke} > 0$ and $\tau_{je} > 0$. Pick indices $y, z$ such that $(A^j)_{yye} > 0$ and $(A^k)_{zze} > 0$. If $y = z$ then for all large $M$ we have $a_{yyMe} > 0$, and because $H_y = G$ we have for all $g$ and all large $M$ that $a_{yyMg} > 0$. It then easily follows from the irreducibility of $\overline{A}$ that $A$ is $G$-primitive.

So suppose $y \neq z$. Because $\overline{A}$ is irreducible, we may choose integers $s, s'$ such that $\overline{A}^s(y, z) > 0$ and $\overline{A}^{s'}(y, z) > 0$. There are corresponding paths $\pi, \pi'$ in the labeled graph with adjacency matrix $A$, say with weights $g$ and $g'$. Let $\pi^*$ be a path from $q$ to $q$ with length $k$ and weight $e$. The concatenation $\pi\pi'$ is a path of length $s + s'$ and weight $gg'$ from $y$ to $y$. Pick $r$ such that $(gg')^r = e$. Then the path $(\pi\pi')^{jr-1}\pi\pi^*\pi'$ is a path from $y$ to $y$ of weight $e$ and length $jr + k$, which is relatively prime to $j$. The argument of the last paragraph then applies to show $A$ is $G$-primitive .

(3) Suppose $G$ is abelian. In this case the conjugate groups $H_i$ are equal and must equal $\cup_k\{g : \tau_{kg} > 0\}$. Thus $A$ is $G$-primitive if and only if for some relatively prime $j, k$ we have $\tau_{ke} \gg 0$ and $\tau_{je} \gg 0$. This is easily checked with $\det(I - tA)$, which constructively determines $(\mathrm{tr}(A^k))_{k\in\mathbb{N}}$.

□

**Corollary 3.8.** *A matrix $A$ over $\mathbb{Z}_+G$ defines a mixing $G$-extension if and only if*

*A is essentially G-primitive .*

*Proof.* The $G$ extension defined by $A$ is a SFT defined by $\widetilde{A}$, and therefore is topologically mixing if and only if $\widetilde{A}$ is essentially primitive as a matrix over $\mathbb{Z}_+$. Therefore the corollary follows from the equivalence of (1) and (2) in Proposition 3.7. $\square$

**Polynomial matrices.** Given $A$ over $t\mathbb{Z}_+G[t]$, we have $\sum_n \mathrm{tr}(A^n) = \sum_n \mathrm{tr}((A^\square)^n))t^n$, and for $G$ abelian $\det(I-A) = \det(I-tA^\square)$. By Theorem 3.7, this data determines whether $A$ is $G$-primitive.

We will need the following consequence of Theorem 3.7.

*Proposition* 3.48. Suppose $A = (a)$ is a $1 \times 1$ matrix over $t\mathbb{Z}G_+[t]$ with $A^\square$ $G$-primitive. Let $(\alpha_k)$ be the sequence of elements from $\mathbb{Z}G$ such that

$$\sum_{j=1}^{\infty} a^j \;=\; \sum_{k=1}^{\infty} \alpha_k t^k \ .$$

Then there is a positive real number $c$ such that

$$\lim_{k\to\infty} \frac{1}{(\lambda_A)^k}\,\alpha_k \;=\; c(g_1+\cdots+g_m) \ .$$

*Proof.* The matrix $A^\square$ is the adjacency matrix of a loop graph $\mathcal{G}$ with base vertex 1. Let $a = \sum_{k,g} a_{kg}gt^k$, with the $a_{kg}$ in $\mathbb{Z}_+$. Then in $\mathcal{G}$, for every positive coefficient $a_{kg}$, there are $a_{kg}$ first return loops to 1 of length $k$ and weight $g$. The return loops to 1 are formed from all concatenations of first return loops. Under concatenation, lengths add and weights multiply. Consequently, for all $k$, $(A^\square)^k(1,1) = \alpha_k$. The proposition is then a consequence of Theorem 3.7. $\square$

In the rest of this section, we check that two standard results for SFTs carry over to $G$-SFTs. The main interest of the next proposition is (1) $\iff$ (2). The proof is an adaptation of the proof of Kim and Roush in the $\mathbb{Z}$ case (see [14, Section7.5] or [71]).

*Proposition* 3.49. Suppose $G$ is a finite group, $\mathcal{S} = \mathbb{Z}_+G$ or $\mathcal{S} = \mathbb{Z}G$, and $A$ and $B$ are square matrices over $\mathcal{S}$. Then the following are equivalent.

1. $A$ and $B$ are SE over $\mathcal{S}$.

2. $A^n$ and $B^n$ are ESSE over $\mathcal{S}$ for all large $n$.

3. $A^n$ and $B^n$ are SE over $\mathcal{S}$ for all large $n$.

4. Let $A$ be $n_1 \times n_1$ and let $B$ be $n_2 \times n_2$. Let $n = \max\{n_1, n_2\}$ and let $m = |G|$. Then there exists $k$ such that $A^k$, $B^k$ are SE over $\mathcal{S}$ and $k \equiv 1 \bmod ((mn)^2)!$ .

*Proof.* Clearly (1) $\implies$ (2) $\implies$ (3) $\implies$ (4). Now, to show (4) $\implies$ (1), assume (4). Then we have $\ell \in \mathbb{N}$, $k \equiv 1 \bmod ((mn)^2)!$ and matrices $U, V$ over $\mathcal{S}$ such that the following hold:

$$(A^k)^\ell = UV \ , \quad (B^k)^\ell = VU \ , \quad A^kU = UB^k \ , \quad B^kV = VA^k \ .$$

For $i \geq n$ and $k \geq n$ define $U_i = A^iU$ and $V_j = B^jV$. Then

$$(A^k)^{\ell+i+j} = U_iV_j \ , \quad (B^k)^{\ell+i+j} = V_jU_i \ , \quad A^kU_i = U_iB^k \ , \quad B^kV_j = V_jA^k \ .$$

Via the map $\mathbb{Z}G \to \mathbb{Z}^m$ discussed earlier, this gives a shift equivalence of matrices over $\mathcal{S}$,

$$(\widetilde{A}^k)^{\ell+i+j} = \widetilde{U}_i\widetilde{V}_j \ , \quad (\widetilde{B}^k)^{\ell+i+j} = \widetilde{V}_j\widetilde{U}_i \ , \quad \widetilde{A}^k\widetilde{U}_i = \widetilde{U}_i\widetilde{B}^k \ , \quad \widetilde{B}^k\widetilde{V}_j = \widetilde{V}_j\widetilde{A}^k \ .$$

Choose $i$ such that $\ell + i + j \equiv 1 \mod ((mn)^2)!$. It suffices to show that the two intertwining equations then hold with $k$ replaced by 1 (as this translates to the equations holding with the $\widetilde{\ }$ decorations removed). Let $r = k(\ell + i + j)$.

Consider the intertwining equation for $U_i$. The matrix $A$ is $mn_1 \times mn_1$, and $\mathbb{C}^{mn_1}$ is the direct sum of the kernel $\mathrm{K}_A$ and the image $W_A$ of $A^{mn}$. Because $i \geq mn$, restricted to K we have $\widetilde{A}\widetilde{U_i} = \widetilde{U_i}\widetilde{B} = 0$ . Also, $U_i$ maps $W_A$ isomorphically to $W_B$, the image of $B^{mn}$. An invariant Jordan subspace of $A$ for eigenvalue $\alpha \neq 0$ is mapped by $U_i$ to an invariant Jordan subspace of $B$ for eigenvalue $\beta \neq 0$, such that $\alpha/\beta$ is a root of unity $\xi$ such that $\xi^r = 1$. Because $\xi$ is in the number field generated by $\alpha$ and $\beta$, $\xi$ is a $q$th root of unity with $q \leq (mn)^2$, and therefore $q$ divides $((mn)^2)!$ . Consequently $\xi^r = \xi$ and $\xi = 1$. It follows that $\widetilde{A}\widetilde{U_i} = \widetilde{U_i}\widetilde{B}$ . The same argument works for the other intertwining equation. $\square$

*Proposition* 3.50. Suppose $G$ is a finite group and $A$ and $B$ are square matrices over $\mathbb{Z}_+G$. Then the following are equivalent.

1. The $G$-SFTs $\sigma_A, \sigma_G$ are eventually conjugate.

2. The matrices $A$, $B$ are SE over $\mathbb{Z}_+G$.

*Proof.* Clearly (2) $\implies$ (1). Also, (2) implies $A^n$ and $B^n$ are SE over $\mathbb{Z}_+G$ for all large $n$, and this implies (1) by Proposition 3.49. $\square$

*Proposition* 3.51. Suppose $A, B$ are $G$-primitive . Then the following are equivalent.

1. $A$ and $B$ are SE over $\mathbb{Z}_+G$.

2. $A$ and $B$ are SE over $\mathbb{Z}G$.

*Proof.* Assuming (2), it suffices to prove (1). We have matrices $U, V$ over $\mathbb{Z}G$ giving the assumed shift equivalence of $A, B$. Then $\widetilde{U}, \widetilde{V}$ give a shift equivalence of $\widetilde{A}, \widetilde{B}$. Perhaps after replacing $U, V$ with $-U, -V$ we have that $U$ takes positive left/right eigenvectors of $\widetilde{A}$ to positive left/right eigenvectors for $\widetilde{B}$, and likewise for $V$. It follows from the spectral gap given by primitivity that for large $k$, the matrices $\widetilde{A}^k U$ and $\widetilde{B}^k V$ are strictly positive. They give an SE over $\mathbb{Z}_+$ of $\widetilde{A}, \widetilde{B}$ and consequently produce an SE over $\mathbb{Z}_+G$ of $A, B$. $\square$

## 3.9 $\mathrm{NK}_1(\mathbb{Z}G)$

Let $\mathcal{R}$ be a ring (always assumed to be unital). In this appendix, we give background on the group $\mathrm{NK}_1(\mathcal{R})$, especially for $\mathcal{R} = \mathbb{Z}G$, with $G$ a finite group.

The first algebraic K group is defined by $\mathrm{K}_1(\mathcal{R}) = \mathrm{GL}(\mathcal{R})/\mathrm{El}(\mathcal{R})$, where $\mathrm{GL}(\mathcal{R}) = \varinjlim \mathrm{GL}_n(\mathcal{R})$ and $\mathrm{El}(\mathcal{R}) = \varinjlim \mathrm{El}_n(\mathcal{R})$, $\mathrm{El}_n(\mathcal{R})$ the elementary matrices of size $n$. If $\mathcal{R}$ is also commutative, then the determinant map $\det : \mathcal{R} \to \mathcal{R}^\times$ is a split surjection, and gives a decomposition $\mathrm{K}_1(\mathcal{R}) \cong \mathrm{SK}_1(\mathcal{R}) \oplus \mathcal{R}^\times$, where $\mathrm{SK}_1(\mathcal{R}) = \ker(\det)$, and $\mathcal{R}^\times$ denotes the group of units in $\mathcal{R}$.

The group $\mathrm{NK}_1(\mathcal{R})$ is defined to be $\ker(\mathrm{K}_1(\mathcal{R}[t]) \overset{t\to 0}{\to} \mathrm{K}_1(\mathcal{R}))$. The exact sequence $0 \to t\mathcal{R}[t] \to \mathcal{R}[t] \overset{t\to 0}{\to} \mathcal{R} \to 0$ is split on the right, giving a decomposition $\mathrm{K}_1(\mathcal{R}[t]) \cong \mathrm{NK}_1(\mathcal{R}) \oplus \mathrm{K}_1(\mathcal{R})$. Higman's trick shows that $\mathrm{NK}_1(\mathcal{R})$ is generated by elements of the form $[I - tN]$, with $N$ nilpotent. If $R$ is reduced (has no non-trivial nilpotents), then one also has $\mathrm{NK}_1(\mathcal{R}) \subset \mathrm{SK}_1(\mathcal{R}[t])$.

For any ring $\mathcal{R}$, $\mathrm{NK}_1(\mathcal{R})$ either is trivial or is not finitely generated as a group.

For many rings $\mathcal{R}$, $\mathrm{NK}_1(\mathcal{R}) = 0$. For any regular Noetherian ring $\mathcal{R}$, $\mathrm{NK}_1(\mathcal{R}) = 0$. For example, a polynomial ring $\mathcal{R}[x_1, \ldots, x_n]$ is regular Noetherian if $\mathcal{R}$ is a field, $\mathbb{Z}$, a Dedekind domain or any ring with finite global dimension. See [18, 19] for all this and more. However, if $G$ is a non-trivial finite group, then $\mathbb{Z}G$ is not regular, and in general the computation of $\mathrm{NK}_1(\mathbb{Z}G)$ is difficult. If $G$ is any finite group of square-free order, then $\mathrm{NK}_1(\mathbb{Z}G) = 0$ [21]. In [22], it is shown that $\mathrm{NK}_1(\mathbb{Z}[\mathbb{Z}/2 \oplus \mathbb{Z}/2])$, $\mathrm{NK}_1(\mathbb{Z}[\mathbb{Z}/4])$, and $\mathrm{NK}_1(\mathbb{Z}[D_4])$, where $D_4$ denotes the dihedral group of order 8, are all non-zero. In fact, both $\mathrm{NK}_1(\mathbb{Z}[\mathbb{Z}/2 \oplus \mathbb{Z}/2])$ and $\mathrm{NK}_1(\mathbb{Z}[\mathbb{Z}/4])$, as abelian groups, are isomorphic to a countably infinite direct sum of copies of $\mathbb{Z}/2$, while $\mathrm{NK}_1(\mathbb{Z}[D_4])$ is a quotient of a direct sum of a countably infinite free $\mathbb{Z}/4$ module and a countably infinite free $\mathbb{Z}/2$ module [22].

While the situation for $\mathbb{Z}[G]$ with $G$ a general finite group is complicated, more is known for finite abelian groups. It follows from Theorem 3.12 in [72] together with Theorem 1.4 from [22] that $NK_1(\mathbb{Z}[\mathbb{Z}/p^n]) \neq 0$ for $n \geq 2$ with $p$ prime[4]. This taken together with Theorem 3.6 in [72] then implies that for a general finite abelian group $G = \bigoplus_{i=1}^{n} \mathbb{Z}/p_i^{k_i}$, $NK_1(\mathbb{Z}[G])$ is non-zero if one of its $p$-primary cyclic components has $p$-rank greater than 1, i.e. $k_i \geq 2$ for some $1 \leq i \leq n$.

For any ring $\mathcal{R}$ and finite group $G$, $\mathrm{NK}_1(\mathcal{R}G)$ is a torsion group [19, 74]. In fact, [74, Theorem A] shows that the order of every element of $\mathrm{NK}_1(\mathcal{R}G)$ is some power of $|G|$, whenever $\mathrm{NK}_1(\mathcal{R}) = 0$. (For $\mathcal{R} = \mathbb{Z}$, and other rings, this is a result of Weibel.) In particular, if $P$ is a finite $p$-group, then every element of $\mathrm{NK}_1(\mathbb{Z}P)$ has $p$-primary order [74].

[4] This is also proved in [73]

*Proposition* 3.52. Suppose the ring $\mathcal{R}$ is commutative and reduced (i.e., has no nonzero nilpotent element). Then the following hold.

1. Let $N$ be a nilpotent matrix over $\mathcal{R}$. Then $\mathrm{tr}(N^k) = 0$ for all $k$ in $\mathbb{N}$.

2. $\mathrm{NK}_1(\mathcal{R}) \subset \mathrm{SK}_1(\mathcal{R}[t])$.

If $G$ is a finitely generated abelian group, then $\mathrm{NK}_1(\mathbb{Z}G) \subset \mathrm{SK}_1(\mathbb{Z}G[t])$.

*Proof.* (1) Suppose $N$ is nilpotent with $\mathrm{tr}(N^\ell) = \alpha \neq 0$. Without loss of generality, suppose $\mathrm{tr}(N^j) = 0$ for $j > \ell$. Set $M = N^\ell$ and suppose $M^J = 0$. Let $\det(I - tM) = 1 - c_1 t - c_2 t^2 - \cdots$. Then $c_1 = \alpha$ and for $k > 1$,

$$\mathrm{tr}(M^k) = kc_k + \sum_{1 \leq j < k} c_j \mathrm{tr}(M^{k-j}) = kc_k + c_{k-1}\mathrm{tr}(M) \ .$$

By induction, $(k!)c_k = (-1)^{k+1}\alpha^k$, for all $k$ in $\mathbb{N}$. Since $\det(I - tM)$ is a polynomial, $\alpha$ is nilpotent, a contradiction.

(2) An element of $\mathrm{NK}_1(\mathcal{R})$ contains a matrix of the form $I - tN$, where $N$ is nilpotent over $\mathcal{R}$. Since $I - tN$ is invertible, $\det(I - tN)$ must be a unit in the polynomial ring $\mathcal{R}[t]$. Because $\mathcal{R}$ is commutative and reduced, the only units in $\mathcal{R}[t]$ are degree zero polynomials, and therefore $\det(I - tN) = 1$.

For a finitely generated abelian group $G$, it follows from a theorem of Sehgal [75, page 176] that $\mathbb{Z}G$ has no nilpotent elements. □

For a ring $\mathcal{R}$, the reduced nil group $\mathrm{Nil}_0(\mathcal{R})$ is an abelian group which may be presented by generators and relations as follows. The generator set is the set of nilpotent matrices. The relations are $A = A \oplus 0$ (where 0 is any square zero matrix and $A$ is nilpotent); $A = U^{-1}AU$ ($A$ nilpotent, $U$ invertible over $\mathcal{R}$); and for any block matrix with $A, B$ square nilpotent,

$$A + B = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix} .$$

An important corresondence in K-theory is that the map $N \mapsto I + tN$ (defined for $N$ nilpotent) induces a well defined isomorphism from $\mathrm{Nil}_0(\mathcal{R})$ to $\mathrm{NK}_1(\mathcal{R})$.

**Explicit examples over $\mathbb{Z}G$**

Below we give some explicit examples of elements in $\mathrm{NK}_1$ of certain integral group rings.

*Example* 3.53. We give a $2 \times 2$ matrix $M$ which represents a nontrivial element of $\mathrm{NK}_1(\mathbb{Z}G)$, for the cyclic group $G = \mathbb{Z}/4\mathbb{Z}$. (The justification in [76] for the example is a nontrivial and computer-assisted exercise in K-theory.) We let $\sigma$ be a generator of $G$ and set

$$M = \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}$$

with

$$a = (1-\sigma^2)(x - 2x^2 + 2x^3 - \sigma + x\sigma + x^2\sigma)$$

$$b = (1-\sigma^2)(1 + 2x - x^2 - x^3 - 2x^4 + \sigma - x\sigma - 2x^2\sigma - 3x^3\sigma + 2x^4\sigma)$$

$$c = (1-\sigma^2)(-1 + 2x - 5x^2 + 7x^3 - 3x^4 + 2x^5 - \sigma + 2x\sigma - 2x^3\sigma + 3x^4\sigma - 2x^5\sigma)$$

$$d = (1-\sigma^2)(2 + x - 2x^2 - 4x^4 - 2x^5 + \sigma - 3x\sigma - x^2\sigma - 4x^3\sigma + 6x^4\sigma - 4x^5\sigma + 4x^6\sigma) .$$

Because entries of the $2 \times 2$ matrix $M$ have maximum degree 6, we can systematialy produce from $M$ a $12 \times 12$ nilpotent matrix $N$ which is nontrivial in $\mathrm{Nil}_0(\mathbb{Z}G)$. We could work harder to reduce the $12 \times 12$ size a bit, but we do not know how to produce a small nilpotent matrix nontrivial in $\mathrm{Nil}_0(\mathbb{Z}G)$.

*Example* 3.54. One could ask for an explicit example of two $G$-primitive matrices over $\mathbb{Z}_+G$ with $G$ abelian which are shift equivalent but not strong shift equivalent over $\mathbb{Z}_+G$ (and thus present nonisomorphic mixing group extensions). We don't know small matrix examples for this, because we don't know small examples of nilpotents nontrivial in $\mathrm{NK}_1(\mathbb{Z}G)$. We can do a bit better with polynomial matrix presentations. With $G = \mathbb{Z}/4\mathbb{Z}$ and $a, b, c, d$ from Example 3.53 and $e, f$ elements of $\mathbb{Z}_+G[x]$, consider the $4 \times 4$ matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e & f & 0 & 0 \\ e & f & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e-2f & f & f & f \\ e-2f+(a+b+c+d) & f & f-(a+c) & f-(b+d) \\ e-(a+b) & 0 & f-c & f-d \\ e-(c+d) & 0 & f-a & f-b \end{pmatrix} := L \ .$$

Let $K = \left(\begin{smallmatrix} e & f \\ e & f \end{smallmatrix}\right)$. Choosing $f$, and then $e$, with sufficiently large coefficients, one has $K$ and $L$ over $\mathbb{Z}_+G[t]$ such that $K^\square$ and $L^\square$ are $G$-primitive matrices. Because $I - K$ and $I - L$ are not $\mathrm{El}(\mathbb{Z}G[t])$ equivalent, $K^\square$ and $L^\square$ are not SSE over $\mathbb{Z}G$, and therefore the associated group extensions cannot be isomorphic. However, $K^\square$ and $L^\square$ are shift equivalent over $\mathbb{Z}G$ and therefore (since they are $G$-primitive) shift equivalent over $\mathbb{Z}_+G$, by 3.51.

# Chapter 4: Strong shift equivalence and the Generalized Spectral Conjecture

## 4.1 Introduction

The purpose of this chapter is to prove the following theorem and explain its context.

**Theorem 4.1.** *Suppose $\mathcal{R}$ is a dense subring of $\mathbb{R}$, $A$ is a primitive matrix over $\mathcal{R}$ and $B$ is a matrix over $\mathcal{R}$ which is shift equivalent over $\mathcal{R}$ to $A$.*

*Then $B$ is strong shift equivalent over $\mathcal{R}$ to a primitive matrix.*

We begin with the context. By ring, we mean a ring with 1; by a semiring, we mean a semiring containing $\{0, 1\}$. A primitive matrix is a square matrix which is nonnegative (meaning entrywise nonnegative) such that for some $k > 0$ its $k$th power is a positive matrix. Definitions and more background for shift equivalence (SE) and strong shift equivalence (SSE) are given in Section 4.2.

We recall the Spectral Conjecture for primitive matrices from [8]. In the statement, $\Delta = (d_1, \ldots, d_k)$ is a $k$-tuple of nonzero complex numbers. $\Delta$ is the *nonzero spectrum* of a matrix $A$ if $A$ has characteristic polynomial of the form $\chi_A(t) = t^m \prod_{1 \le i \le k}(t - d_i)$. $\Delta$ has a *Perron value* if there exists $i$ such that $d_i > |d_j|$

when $j \neq i$. The *trace* of $\Delta$ is $\mathrm{tr}(\Delta) = d_1 + \cdots + d_k$. $\Delta^n$ denotes $((d_1)^n, \ldots, (d_k)^n)$, the tuple of $n$th powers; and the *nth net trace* of $\Delta$ is

$$\mathrm{tr}_n(\Delta) = \sum_{d|n} \mu(n/d)\mathrm{tr}(\Delta^d)$$

in which $\mu$ is the Möbius function ($\mu(1) = 1$; $\mu(n) = (-1)^r$ if $n$ is the product of $r$ distinct primes; $\mu(n) = 0$ if $n$ is divisible by the square of a prime).

**Spectral Conjecture 4.2.** *[8] Let $\mathcal{R}$ be a subring of $\mathbb{R}$. Then $\Delta$ is the nonzero spectrum of some primitive matrix over $\mathcal{R}$ if and only if the following conditions hold:*

1. *$\Delta$ has a Perron value.*
2. *The coefficients of the polynomial $\prod_{i=1}^{k}(t - d_i)$ lie in $\mathcal{R}$.*
3. *If $\mathcal{R} = \mathbb{Z}$, then for all positive integers $n$, $\mathrm{tr}_n(\Delta) \geq 0$;*
   *if $\mathcal{R} \neq \mathbb{Z}$, then for all positive integers $n$ and $k$,*
   *(i) $\mathrm{tr}(\Delta^n) \geq 0$ and (ii) $\mathrm{tr}(\Delta^n) > 0$ implies $\mathrm{tr}(\Delta^{nk}) > 0$.*

It is not difficult to check that the nonzero spectrum of a primitive matrix satisfies the three conditions [8]. (We remark, following [77] it is known that the nonzero spectra of symmetric primitive matrices cannot possibly have such a simple characterization.)

To understand the possible spectra of nonnegative matrices is a classical problem of linear algebra (for early background see e.g. [8]) on which interesting progress continues (see e.g. [78–81] and their references). Understanding the nonzero spectra of primitive matrices is a variant of this problem and also an approach to it: to know the minimal size of a primitive matrix with a prescribed nonzero spectrum is to solve the classical problem (for details, see [8]); and it is in the primitive case that the Perron-Frobenius constraints manifest most simply.

Finally, as the spectra of matrices over various subrings of $\mathbb{R}$ appear in applications, in which the nonzero part of the spectrum is sometimes the relevant part [8, 10], it is natural to consider the nonzero spectra of matrices over arbitrary subrings of $\mathbb{R}$.

The Spectral Conjecture has been proved in enough cases that it seems almost certain to be true in general. For example, it is true under any of the following conditions:

- The Perron value of $\Lambda$ is in $\mathcal{R}$ (this always holds when $\mathcal{R} = \mathbb{R}$) or is a quadratic integer over $\mathcal{R}$ [8].
- $\mathrm{tr}(\Lambda) > 0$ [8, Appendix 4]
- $\mathcal{R} = \mathbb{Z}$ or $\mathbb{Q}$ [9].

The general proofs in [8] do not give even remotely effective general bounds on the size of a primitive matrix realizing a given nonzero spectrum. The methods used in [9] for the case $\mathcal{R} = \mathbb{Z}$ are much more tractable but still very complicated. However, there is now an elegant construction of Tom Laffey [79] which proves the conjecture for $\mathcal{R} = \mathbb{R}$ in the central special case of positive trace, and in some other cases; where it applies, the construction provides meaningful bounds on the size of the realizing matrix in terms of the spectral gap.

The nonzero spectrum of a matrix is a "stable" or "eventual" invariant of a matrix. For a matrix over a field, an obvious finer invariant is the isomorphism class of the nonnilpotent part of its action as a linear transformation. The classification of matrices over a field by this invariant is the same as the classification up to shift equivalence over the field; for matrices over general rings, from the module viewpoint (see Sec.4.2), shift equivalence is the natural generalization of the isomorphism class of this nonnilpotent linear transformation. For some rings, an even finer invariant is the strong shift equivalence class. The Generalized Spectral Conjecture of Boyle and Handelman (in both forms below) heuristically is saying that only the obvious necessary spectral conditions constrain the eventual algebra of a primitive matrix over a subring of $\mathbb{R}$, regardless of the subring under consideration.

**Generalized Spectral Conjecture (weak form, 1991) 4.3.** *Suppose $\mathcal{R}$ is a subring of $R$ and $A$ is a square matrix over $\mathcal{R}$ whose nonzero spectrum satisfies the three necessary conditions of the Spectral Conjecture. Then $A$ is SE over $\mathcal{R}$ to a primitive matrix.*

**Generalized Spectral Conjecture (strong form, 1993) 4.4.** *Suppose $\mathcal{R}$ is a subring of $R$ and $A$ is a square matrix over $\mathcal{R}$ whose nonzero spectrum satisfies the three necessary conditions of the Spectral Conjecture. Then $A$ is SSE over $\mathcal{R}$ to a primitive matrix.*

The weak form was stated in [8, p.253] and [3, p.124]. The strong form was stated in [10, Sec. 8.4]), along with an explicit admission that the authors of the conjecture did not know if the conjectures were equivalent (not knowing if shift equivalence over a ring implies strong shift equivalence over it). Following [45] (see Theorem 4.5), we know now that the strong form of the Generalized Spectral Conjecture was not a vacuous generalization: there are subrings of $\mathbb{R}$ over which SE does not imply SSE (Example 4.4). The results of [45] also provide enough structure that we can prove Theorem 4.1, which shows that the two forms of the Generalized Spectral Conjecture are equivalent.

**Note!** In contrast to the statement of the Generalized Spectral Conjecture for *primitive* matrices, it is *not* the case that the existence of a strong shift equivalence over $\mathcal{R}$ from a matrix $A$ over $\mathcal{R}$ to a nonnegative matrix can in general be characterized by a spectral condition on $A$. There are dense subrings of $\mathbb{R}$ over which there are nilpotent matrices which are not SSE to nonnegative matrices (Remark 4.5).

There is some motivation from symbolic dynamics for pursuing the zero trace case of the GSC. The Kim-Roush and Wagoner primitive matrix counterexamples [37, 82] to Williams' conjecture SE-$\mathbb{Z}_+ \implies$ SSE-$\mathbb{Z}_+$ rely absolutely on certain zero-positive patterns of traces of powers of the given matrix. We still do not know whether the refinement of SE-$\mathbb{Z}_+$ by SSE-$\mathbb{Z}_+$ is algorithmically undecidable or (at another extreme) if it allows some finite description involving such sign patterns. We are looking for any related insight.

## 4.2 Shift equivalence and strong shift equivalence

Suppose $\mathcal{R}$ is a subset of a semiring and $\mathcal{R}$ contains $\{0, 1\}$. (For example, $\mathcal{R}$ could be $\mathbb{Z}, \mathbb{Z}_+, \{0, 1\}, \mathbb{R}, \mathbb{R}_+, \ \dots$ ) Square matrices $A, B$ over $\mathcal{R}$ (not necessarily of the same size) are *elementary strong shift equivalent over* $\mathcal{R}$ (ESSE-$\mathcal{R}$) if there exist matrices $U, V$ over $\mathcal{R}$ such that $A = UV$ and $B = VU$. Matrices $A, B$ are *strong shift equivalent over* $\mathcal{R}$ (SSE-$\mathcal{R}$) if there are a positive integer $\ell$ (the *lag* of the given SSE) and matrices $A = A_0, A_1, \dots, A_\ell = B$ such that $A_{i-1}$ and $A_i$ are ESSE-$\mathcal{R}$, for $1 \le i \le \ell$. For matrices over a subring of $\mathbb{R}$, the relation ESSE-$\mathcal{R}$ is never transitive. For example, if matrices $A, B$ are ESSE over $\mathbb{R}$, $j > 1$ and $A^j \neq 0$, then $B^{j-1} \neq 0$; but if $A$ is the $n \times n$ matrix such that $A(i, i+1) = 0$ for $1 \le i < n$ and $A = 0$ otherwise, then $A$ is SSE-$\mathcal{R}$ to $(0)$. Over any ring $\mathcal{R}$, the relation SSE-$\mathcal{R}$ on square matrices is generated by similarity over $\mathcal{R}$ ($U^{-1}AU \sim A$) and nilpotent extensions, $\left(\begin{smallmatrix} A & X \\ 0 & 0 \end{smallmatrix}\right) \sim A \sim \left(\begin{smallmatrix} 0 & X \\ 0 & A \end{smallmatrix}\right)$ [26].

Square matrices $A, B$ over $\mathcal{R}$ are *shift equivalent over* $\mathcal{R}$ (SE-$\mathcal{R}$) if there exist a positive integer $\ell$ and matrices $U, V$ over $\mathcal{R}$ such that the following hold:

$$\begin{aligned} A^\ell &= UV & \qquad B^\ell &= VU \\ AU &= UB & \qquad BV &= VA\ . \end{aligned}$$

Herem $\ell$ is the *lag* of the given SE. It is always the case that SSE over $\mathcal{R}$ implies SE over $\mathcal{R}$: from a given lag $\ell$ SSE one easily creates a lag $\ell$ SE [1]. For certain semirings $\mathcal{R}$, including above all $\mathcal{R} = \mathbb{Z}_+$, the relations of SSE and SE over $\mathcal{R}$ are significant for symbolic dynamics. The relations were introduced by Williams for the cases $\mathcal{R} = \mathbb{Z}_+$ and $\mathcal{R} = \{0, 1\}$ to study the classification of shifts of finite type. Matrices over $\mathbb{Z}_+$ are SSE over $\mathbb{Z}_+$ if and only if they define topologically conjugate shifts of finite type. However, the relation SSE-$\mathbb{Z}_+$ to this day remains mysterious and is not even know to be decidable. In contrast, SE-$\mathbb{Z}_+$ is a tractable, decidable, useful and very strong invariant of SSE-$\mathbb{Z}_+$.

Suppose now $\mathcal{R}$ is a ring, and $A$ is $n \times n$ over $\mathcal{R}$. T o see the shift equivalence relation SE-$\mathcal{R}$ more conceptually, recall that the direct limit $G_A$ of $\mathcal{R}^n$ under the $\mathcal{R}$-module homomorphism $x \mapsto Ax$ is the set of equivalence classes $[x, k]$ for $x \in \mathcal{R}^n, k \in \mathbb{Z}_+$ under the equivalence relation $[x, k] \sim [y, j]$ if there exists $\ell > 0$ such that $A^{j+\ell}x = A^{k+\ell}y$. $G_A$ has a well defined group structure ($[x, k] + [y, j] = [A^k x + A^j y, j + k]$) and is an $\mathcal{R}$-module ($r : [x, k] \mapsto [xr, k]$). $A$ induces an $\mathcal{R}$-module isomorphism $\hat{A} : [x, k] \mapsto [Ax, k]$ with inverse $[x, k] \mapsto [x, k + 1]$. $G_A$ becomes an $\mathcal{R}[t]$ module (also an $\mathcal{R}[t, t^{-1}]$ module) with $t : [x, k] \mapsto [x, k+1]$. $A$ and $B$ are SE-$\mathcal{R}$ if and only if these $\mathcal{R}[t]$-modules are isomorphic (equivalently, if and only if they are isomorphic as $\mathcal{R}[t, t^{-1}]$ modules). If the square matrix $A$ is $n \times n$, then $I - tA$ defines a homomorphism $\mathcal{R}^n \to \mathcal{R}^n$ by the usual multiplication $v \mapsto (I - tA)v$, and $\mathrm{cok}(I - tA)$ is an $\mathcal{R}[t]$-module which is isomorphic to the $\mathcal{R}[t]$-module $G_A$. For more detail and references on these relations (by no means original to us) see [14, 45].

Williams introduced SE and SSE in the 1973 paper [1]. For any principal ideal domain $\mathcal{R}$, Effros showed SE-$\mathcal{R}$ implies SSE-$\mathcal{R}$ in the 1981 monograph [2] (see [16] for Williams' proof for the case $\mathcal{R} = \mathbb{Z}$). In the 1993 paper [3], Boyle and Handelman extended this result to the case that $\mathcal{R}$ is a Dedekind domain (or, a little more generally, a Prüfer domain). Otherwise, the relationship of SE and SSE of matrices over a ring remained open until the recent paper [45], which explains the relationship in general as follows.

**Theorem 4.5.** *[45] Suppose $A, B$ are SE over a ring $\mathcal{R}$.*

1. *There is a nilpotent matrix $N$ over $\mathcal{R}$ such that $B$ is SSE over $\mathcal{R}$ to the matrix* $A \oplus N = \begin{pmatrix} A & 0 \\ 0 & N \end{pmatrix}$.
2. *The map $[I - tN] \to [A \oplus N]_{SSE}$ induces a bijection from $\mathrm{NK}_1(\mathcal{R})$ to the set of SSE classes of matrices over $\mathcal{R}$ which are in the SE-$\mathcal{R}$ class of $A$.*

We will say just a little now about $\mathrm{NK}_1(\mathcal{R})$, a group of great importance in algebraic $K$-theory; for more background, we have found [18, 19, 25] very helpful. $\mathrm{NK}_1(\mathcal{R})$ is the kernel of the map $K_1(\mathcal{R}[t]) \to K_1(\mathcal{R})$ induced by the ring homomorphism $\mathcal{R}[t] \to \mathcal{R}$ which sends $t$ to 0. The finite matrix $I - tN$ corresponds to the matrix $I - (tN)_\infty$ in the group $\mathrm{GL}(\mathcal{R}[t])$ (with $I$ denoting the $\mathbb{N} \times \mathbb{N}$ identity matrix and $(tN)_\infty$ the $\mathbb{N} \times \mathbb{N}$ matrix which agrees with $tN$ in an upper left corner and is otherwise zero). Every class of $\mathrm{NK}_1(\mathcal{R})$ contains a matrix of the form $I - (tN)_\infty$ with $N$ nilpotent over $\mathcal{R}$. $\mathrm{NK}_1(\mathcal{R})$ is trivial for many rings (e.g., any field, or more generally any left regular Noetherian ring) but not for all rings. If $\mathrm{NK}_1(\mathcal{R})$ is not trivial, then it is not finitely generated as a group. From the established theory, it is easy to give an example of a subring $\mathcal{R}$ of $\mathbb{R}$ for which $\mathrm{NK}_1(\mathcal{R})$ is not trivial (Example 4.4).

## 4.3 Proof of Theorem 4.1

*Proof of Theorem 4.1.* Given a square matrix $M$ over $\mathbb{R}$, let $\lambda_M$ denote its spectral radius and define the matrix $|M|$ by $|M|(i,j) = |M(i,j)|$.

By Theorem 4.5, let $N$ be a nilpotent matrix such that $B$ is SSE over $\mathcal{R}$ to the matrix $\begin{pmatrix} A & 0 \\ 0 & N \end{pmatrix}$ . Suppose $M$ is a matrix SSE over $\mathcal{R}$ to $N$ and $M$ also satisfies the following conditions:

1. $\lambda_{|3M|} < \lambda_A$
2. For all positive integers $n$, $\mathrm{trace}(|3M|^n) \leq \mathrm{trace}(A^n)$ .
3. For all positive integers $n$ and $k$, if $\mathrm{tr}(|3M|^n) < \mathrm{tr}(A^n)$, then $\mathrm{tr}(|3M|^{nk}) < \mathrm{tr}(A^{nk})$.

Then by the Submatrix Theorem (Theorem 3.1 of [8]), there is a primitive matrix $C$ SSE over $\mathcal{R}$ to $A$ such that $|3M|$ is a proper principal submatrix of $C$. Without loss of generality, let this submatrix occupy the upper left corner of $C$. Define $M_0$ to be the matrix of size matching $C$ which is $M$ in its upper left corner and which is zero in other entries. Then $B$ is SSE over $\mathcal{R}$ to the matrix $\begin{pmatrix} C & 0 \\ 0 & M_0 \end{pmatrix}$. Choose $\epsilon \in \mathcal{R}$ such that $1/3 < \epsilon < 2/3$ and compute

$$\begin{pmatrix} I & -\epsilon I \\ 0 & I \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & M_0 \end{pmatrix} \begin{pmatrix} I & \epsilon I \\ 0 & I \end{pmatrix} = \begin{pmatrix} C & \epsilon(C - M_0) \\ 0 & M_0 \end{pmatrix}$$

$$\begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \begin{pmatrix} C & \epsilon(C - M_0) \\ 0 & M_0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} = \begin{pmatrix} (1-\epsilon)C + \epsilon M_0 & \epsilon(C - M_0) \\ (1-\epsilon)(C - M_0) & \epsilon C + (1-\epsilon) M_0 \end{pmatrix} := G \ .$$

The matrix $G$ is SSE over $\mathcal{R}$ to $B$, and it is nonnegative. The diagonal blocks have positive entries wherever $C$ does; because $C$ is primitive, there is a $j > 0$ such that $C^j > 0$, and therefore the diagonal blocks of $G^j$ are also positive. Because neither offdiagonal block of $G$ is the zero block, it follows that $G$ is primitive.

So, it suffices to find $M$ SSE over $\mathcal{R}$ to $N$ satisfying the conditions (1)-(3) above. Choose $K$ such that $\mathrm{tr}(A^k) > 0$ for all $k > K$. Let $n$ be the integer such that $N$ is $n \times n$, and let $J$ by the integer provided by Proposition 4.3 given $n$ and $K$. Given this $J$, choose $\epsilon > 0$ such that for any $J \times J$ matrix $M$ with $||M||_\infty < \epsilon$, we have $\lambda_{3|M|} < \lambda_A$ and for $k > K$ we also have $\mathrm{tr}(|3M|^k) < \mathrm{tr}(A^k)$. Now let $\delta > 0$ be as provided by Proposition 4.3 for this $\epsilon$.

If we can now find an $n \times n$ nilpotent matrix $N'$ which is SSE over $\mathcal{R}$ to $N$ and satisfies $||N'|| < \delta$, then we can apply Proposition 4.3 to this $N'$ to produce a matrix $M$ SSE over $\mathcal{R}$ to $N$ and with $||M|| < \epsilon$ and with $\mathrm{tr}(M^k) = 0$ for $1 \leq k \leq K$. This matrix $M$ will satisfy the conditions (1)-(3).

Pick $\gamma > 0$ such that $||\gamma N||_\infty < \delta$. There is a matrix $U$ in $\mathrm{SL}(n, \mathbb{R})$ such that $U^{-1}NU = \gamma N$. The matrix $U$ is a product of basic elementary matrices over $\mathbb{R}$, and these can be approximated arbitrarily closely by basic elementary matrices over $\mathcal{R}$. Consequently there is a matrix $V$ in $\mathrm{SL}(n, \mathcal{R})$ such that $||V^{-1}NV||_\infty < \delta$. Choose $N' = V^{-1}NV$. □

To prove the Proposition 4.3 on which the proof of Theorem 4.1 depends, we use a correspondence proved in [45]. We need some definitions.

Given a finite matrix $A$, let $A_\infty$ denote the $\mathbb{N} \times \mathbb{N}$ matrix which has $A$ as its upper left corner and is otherwise zero. In any $\mathbb{N} \times \mathbb{N}$ matrix, $I$ denotes the infinite identity matrix. Given a ring $R$, $\mathrm{El}(R)$ is the group of $\mathbb{N} \times \mathbb{N}$ matrices over $R[t]$, equal to the infinite identity matrix except in finitely many entries, which are products of basic elementary matrices (these basic matrices are by definition equal to $I$ except perhaps in a single offdiagonal entry). For finite matrices $A, B$, the matrices $I - A_\infty$ and $I - B_\infty$ are $\mathrm{El}(R[t])$ equivalent if there are matrices $U, V$ in $\mathrm{El}(R[t])$ such that $U(I - A_\infty)V = I - B_\infty$.

**Definition 4.1.** Given a finite matrix $A$ over $t\mathcal{R}[t]$, choose $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $A_1, \ldots A_k$ are $n \times n$ matrices over $\mathcal{R}$ such that

$$A_\infty = \sum_{i=1}^{k} t^i (A_i)_\infty$$

and define a finite matrix $\mathcal{A}^\square = \mathcal{A}^{\sharp(k,n)}$ over $\mathcal{R}$ by the following block form, in which every block is $n \times n$:

$$\mathcal{A}^\square = \begin{pmatrix} A_1 & A_2 & A_3 & \ldots & A_{k-2} & A_{k-1} & A_k \\ I & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & I & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & I & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & I & 0 \end{pmatrix}.$$

In the definition, there is some freedom in the choice of $\mathcal{A}^\square$: $k$ can be increased by using zero matrices, and $n$ can be increased by filling additional entries of the $A_i$ with zero. These choices do not affect the SSE-$\mathcal{R}$ class of $\mathcal{A}^\square$.

**Theorem 4.6.** *[45] Let $\mathcal{R}$ be a ring. Then there is a bijection between the following sets:*

- *the set of $\mathrm{El}(\mathcal{R}[t])$ equivalence classes of $\mathbb{N} \times \mathbb{N}$ matrices $I - A_\infty$ such that $A$ is a finite matrix over $t\mathcal{R}[t]$*

- *the set of SSE-$\mathcal{R}$ classes of square matrices over $\mathcal{R}$.*

*The bijection from* $\mathrm{El}(\mathcal{R}[t])$ *equivalence classes to SSE-$\mathcal{R}$ classes is induced by the map $I - A_\infty \mapsto \mathcal{A}^\square$. The inverse map (from the set of SSE-$\mathcal{R}$ classes) is induced by the map sending $A$ over $\mathcal{R}$ to the* $\mathbb{N} \times \mathbb{N}$ *matrix $I - tA$.*

By the degree of a matrix with polynomial entries we mean the maximum degree of its entries. If $M$ is a matrix over $\mathbb{R}[t]$, with entries $M(i,j) = \sum_{i,j,k} m_{ijk} t^k$, then we define $||M|| = \max_{k>0} \max_{i,j} |m_{ijk}|$. If $M$ is a matrix over $\mathbb{R}$, with $M(i,j) = m_{ij}$, then $||M||_\infty$ is the usual sup norm, $||M||_\infty = \max_{i,j} |m_{ij}|$.

**Lemma 4.2.** Suppose $\mathcal{R}$ is a dense subring of $\mathbb{R}$ and $A$ is an $n \times n$ matrix of degree $d$ over $t^k\mathcal{R}[t]$, with entries $a_{ij} = \sum_{1 \le r \le d} a_{ij}^{(r)} t^r$. Suppose $\sum_{i=1}^n a_{ii}^{(k)} = 0$ and $||A|| \le \frac{1}{4n^2}$. Then there is an $n \times n$ matrix $B$ over $t^{k+1}\mathcal{R}[t]$ such that $I - A_\infty$ is $\mathrm{El}(\mathcal{R}[t])$ equivalent to $I - B_\infty$ and the following hold:

1. $\mathrm{degree}(B) \;\le\; \mathrm{degree}(A) + 3k$.

2. $||B|| \;\le\; 4n^3 ||A||$ .

*Proof.* For finite square matrices $I - C$ and $I - D$, we use $I - C \sim I - D$ to denote elementary equivalence over $\mathcal{R}[t]$ of $I - C_\infty$ and $I - D_\infty$. We have

$$I - A \;=\; \begin{pmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 1-a_{22} & \cdots & -a_{2n} \\ \vdots & & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1n} & a_{11}^{(k)}t^k \\ -a_{21} & 1-a_{22} & \cdots & -a_{2n} & a_{22}^{(k)}t^k \\ \vdots & & \ddots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} & a_{nn}^{(k)}t^k \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} := I - A_1 \; .$$

In order, apply the following elementary operations:

1. For $1 \le j \le n$, add column $n+1$ to column $j$ of $I - A_1$, to produce a matrix $I - A_2$. Then $\mathrm{degree}(A_2) = \mathrm{degree}(A)$; the diagonal entries of $A_2$ lie in $t^{k+1}\mathcal{R}[t]$; and $||A_2|| \le 2||A_1|| = 2||A||$. Every entry in row $n+1$ of $I - A_2$ equals 1. (By definition these entries have no impact on $||A_2||$.)

2. For $1 \le i \le n$, add (-1)(row $i$) of $(I - A_2)$ to row $n+1$ to form $I - A_3$. Then the entries of $A_3$ lie in $t^k\mathcal{R}[t]$, and the diagonal entries of $A_3$ still lie in $t^{k+1}\mathcal{R}[t]$, since $\sum_{i=1}^n a_{ii}^{(k)} = 0$ . We have $||A_3|| \le n||A_2|| \le 2n||A|| < 1$ and $\mathrm{degree}(A_3) \le \mathrm{degree}(A)$ .

3. For $1 \le i \le n$, add $(-a_{ii}^{(k)}t^k)$(row $n+1$) of $(I - A_3)$ to row $i$ to form $I - A_4$.

In block form,

$$I - A_4 \;=\; \begin{pmatrix} I - A_5 & 0 \\ \\ x & 1 \end{pmatrix}$$

in which $A_5$ is $n \times n$ and $x = (x_1 \cdots x_n)$. Adding multiples of column $n+1$ to columns $1, \ldots, n$ to clear out $x$, we see $I - A_5 \sim I - A$. We have degree$(A_5) \leq$ degree$(A) + k$ and

$$\begin{aligned} ||A_5|| \;&\leq\; ||A_3|| + (||A||)(||A_3||) \\ &\leq\; 2||A_3|| \;\leq\; 4n||A|| \;<\; 1\ . \end{aligned}$$

In $A_5$, the diagonal terms lie in $t^{k+1}\mathcal{R}[t]$ and the offdiagonal terms lie in $t^k\mathcal{R}[t]$. In the next two steps, we apply elementary operations to clear the degree $k$ terms outside the diagonal. We use part of a clearing algorithm from [61].

4. Let $b_{ij}$ be the coefficient of $t^k$ in $A_5(i,j)$. For $2 \leq i \leq n$, add $(-b_{1j}t^k)$(row $j$) to row 1. Continuing in order for rows $i = 2, \ldots, n-1$: for $i+1 \leq j \leq n$, add $(-b_{ij}t^k)$(row $j$) to row $i$. Let $(I - A_6)$ be the resulting matrix. The entries of $A_6$ on and above the diagonal lie in $t^{k+1}\mathcal{R}[t]$. We have

$$\text{degree}(A_6) \;\leq\; \text{degree}(A_5) + k \;\leq\; \text{degree}(A) + 2k$$

and

$$\begin{aligned} ||A_6|| \;&\leq\; ||A_5|| + (n-1)||A_5||^2 \\ &\leq\; n||A_5|| \;\leq\; 4n^2||A|| \leq 1\ . \end{aligned}$$

5. Let $c_{ij}$ denote the coefficient of $t^k$ in $A_6(i,j)$. For $2 \leq j \leq n$, add $(-c_{j1}t^k)$(column $j$) of $A_6$ to column 1. Continuing in order for columns $i = 2, \ldots, n-1$: for

$i+1 \le j \le n$, add $(-c_{ji})$(column $j$) to column $i$. For the resulting matrix $(I-B)$, the entries of $B$ lie in $t^{k+1}\mathcal{R}[t]$, with

$$\text{degree}(B) \;\le\; \text{degree}(A_6) + k \;\le\; \text{degree}(A) + 3k$$

and

$$\begin{aligned} ||B|| \;&\le\; ||A_6|| + (n-1)||A_6||^2 \\ &\le\; n||A_6|| \;\le\; 4n^3||A|| \;. \end{aligned}$$

□

*Proposition* 4.3. Suppose $\mathcal{R}$ is a dense subring of $\mathbb{R}$, $n \in \mathbb{N}$ and $K \in \mathbb{N}$. Then there is a $J$ in $\mathbb{N}$ such that for any $\epsilon > 0$ there exists $\delta > 0$ such that the following holds: if $N$ is a nilpotent $n \times n$ matrix over $\mathcal{R}$ and $||N||_\infty < \delta$, then there is a $J \times J$ matrix $M$ over $\mathcal{R}$ such that

1. $M$ is SSE over $\mathcal{R}$ to $N$,
2. $\text{tr}\,(|M|^k) = 0$ for $1 \le k \le K$, and
3. $||M||_\infty < \epsilon$ .

*Proof.* Because $N$ is nilpotent, $\text{tr}(N^k) = 0$ for all positive integers $k$. Set $B_0 = tN$. We define matrices $B_1, \ldots, B_K$ recursively, letting $I - B_{k+1}$ be the matrix $I - B$ provided by Lemma 4.2 from input $I - A = I - B_k$. The conditions of the lemma are satisfied recursively, because the (zero) trace of the $k$th power of the nilpotent matrix $(B_k)^\sharp$ must be (in the terminology of the lemma) $\sum_i a_{ii}^{(k)}$. The matrix $B_K$ is

$n \times n$ with entries of degree at most

$$d := 1 + 3(1) + 3(2) + \cdots + 3(K) = 1 + 3K(K+1)/2 \ .$$

Let $(B_K)_i$ be the matrices, $1 \leq i \leq d$, such that $B_K = \sum_{i=1}^{d} (B_K)_i t^i$ . Define $M$ to be the matrix $(B_K)^\sharp$, an $nd \times nd$ matrix over $\mathcal{R}$ which is SSE over $\mathcal{R}$ to $N$. Set $J = nd$.

It is now clear from condition (2) of Lemma 4.2 and induction that given $\epsilon > 0$, there is a $\delta > 0$ such that $||N|| < \delta$ implies $||(B_K)|| < \epsilon$. (We are not trying to optimize estimates.) With $K > 1$ (without loss of generality), we have $||B_K|| = ||(B_K)^\sharp||_\infty$. This finishes the proof. $\square$

*Example* 4.4. There are subrings of $\mathbb{R}$ with nontrivial $\mathrm{NK}_1$. For example, let $\mathcal{R} = \mathbb{Q}[t^2, t^3, z, z^{-1}]$. By the Bass-Heller-Swan Theorem (see [18], 3.2.22) for any ring $\mathcal{S}$, there is a splitting $K_1(\mathcal{S}[z, z^{-1}]) \cong K_1(\mathcal{S}) \oplus K_0(\mathcal{S}) \oplus \mathrm{NK}_1(\mathcal{S}) \oplus \mathrm{NK}_1(\mathcal{S})$, which implies $\mathrm{NK}_1(\mathcal{S}[z, z^{-1}])$ always contains a copy of $\mathrm{NK}_0(\mathcal{S})$. An elementary argument (see for example exercise 3.2.24 in [18]) shows that $\mathrm{NK}_0(\mathbb{Q}[t^2, t^3]) \neq 0$, so $\mathrm{NK}_1(\mathbb{Q}[t^2, t^3, z, z^{-1}])$ is non-zero. Since $\mathbb{Q}[t^2, t^3, z, z^{-1}]$ can be realized as a subring of $\mathbb{R}$ (by an embedding sending $t, z$ to algebraically independent transcendentals in $\mathbb{R}$) this provides an example of a subring $\mathcal{R}$ of $\mathbb{R}$ for which $\mathrm{NK}_1(\mathcal{R})$ is not zero, and therefore shift equivalence over $\mathcal{R}$ does not imply strong shift equivalence over $\mathcal{R}$.

It is possible to produce explicit examples by tracking through the exact sequences behind the argument of the last paragraph. This is done in [76], and for

$\mathcal{R} = \mathbb{Q}[t^2, t^3, z, z^{-1}]$ yields the following matrix over $\mathcal{R}[s]$,

$$I - M = \begin{pmatrix} 1 - (1 - z^{-1})s^4t^4 & (z-1)(s^2t^2 - s^3t^3) \\ (1 - z^{-1})(s^2t^2)(1 + st + s^2t^2 + s^3t^3) & 1 + (z-1)(s^4t^4) \end{pmatrix},$$

which is nontrivial as an element of $\mathrm{NK}_1(\mathcal{R})$. Writing $M$ as

$$M = \begin{pmatrix} (1 - z^{-1})s^4t^4 & (1-z)(s^2t^2 - s^3t^3) \\ (z^{-1} - 1)(s^2t^2)(1 + st + s^2t^2 + s^3t^3) & (1-z)(s^4t^4) \end{pmatrix} = \sum_{i=1}^{5} s^i M_i$$

with the $M_i$ over $\mathcal{R}$, we obtain (see [45]) a nilpotent matrix $N$ over $\mathcal{R}$,

$$N = \begin{pmatrix} M_1 & M_2 & M_3 & M_4 & M_5 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & (1-z)t^2 & 0 & (1-z)(-t^3) & (1-z^{-1})t^4 & 0 & 0 & 0 \\ 0 & 0 & (z^{-1}-1)t^2 & 0 & (z^{-1}-1)t^3 & 0 & (z^{-1}-1)t^4 & (1-z)t^4 & (z^{-1}-1)t^5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

which is nontrivial as an element of $\mathrm{Nil}_0(\mathcal{R})$, as is the matrix $N'$ obtained by removing the last row and the last column from $N$.

The matrix $N'$ is $9 \times 9$. We don't have a smaller example, and we don't have a decent example of two positive matrices which are shift equivalent but not strong shift equivalent over a subring of $\mathbb{R}$.

*Remark* 4.5. Suppose $\mathcal{R}$ is a subring of $\mathbb{R}$ and $N$ is a nonnegative nilpotent matrix over $\mathcal{R}$. Then there is a permutation matrix $P$ such that $P^{-1}NP$ is triangular with zero diagonal. Using elementary SSEs of the block form

$$\begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{pmatrix} X & Y \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} X \end{pmatrix} = \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix}$$

we see that $P^{-1}NP$ (and hence $N$) is SSE over $\mathcal{R}$ to $[0]$. By Theorem 4.5, with $A = 0$, it follows that a nilpotent matrix $N$ is SSE over $\mathcal{R}$ to a nonnegative matrix if and only if $[I - tN_\infty]$ is trivial in $\mathrm{NK}_1(\mathcal{R})$. Therefore, if (and only if) $\mathrm{NK}_1(\mathcal{R})$ is nontrivial, there will be nilpotent matrices over $\mathcal{R}$ which cannot be SSE over $\mathcal{R}$ to a nonnegative matrix. The matrix $N$ in Example 4.4 is one such example.

## 4.4 Reflections on the Generalized Spectral Conjecture

Is the Generalized Spectral Conjecture true?

For $\mathcal{R} = \mathbb{Z}$, the Spectral Conjecture is true [9]. The GSC is true for $\mathcal{R} = \mathbb{Z}$ for a given $\Delta$ if every entry of $\Delta$ is a rational integer [3]. There is not much more direct evidence for the GSC for $\mathcal{R} = \mathbb{Z}$, but we know of no results which cast doubt.

From here, suppose $\mathcal{R}$ is a dense subring of $\mathbb{R}$. As noted earlier, the Spectral

Conjecture is almost surely true. Theorem 4.1 removes the possibility that the very subtle algebraic invariants following from Theorem 4.5 could be an obstruction to the GSC. The GSC was proved in [3] in the following cases:

1. when the nonzero spectrum is contained in $\mathcal{R}$, and $\mathcal{R}$ is a Dedekind domain with a nontrivial unit;

2. when the nonzero spectrum has positive trace and either (i) the spectrum is real or (ii) the minimal and characteristic polynomials of the given matrix are equal up to a power of the indeterminate.

The following Proposition (almost explicit in [8, Appendix 4]) is more evidence for the GSC in the positive trace case.

*Proposition* 4.6. Suppose the Generalized Spectral Conjecture holds for matrices of positive trace for the ring $\mathbb{R}$. Then it holds for matrices of positive trace for every dense subring $\mathcal{R}$ of $\mathbb{R}$.

*Proof.* Let $A$ be a square matrix over $\mathcal{R}$ of positive trace which over $\mathbb{R}$ is SSE to a primitive real matrix $B$. We need to show that $A$ is SSE over $\mathcal{R}$ to a primitive matrix.

By [83] (or the alternate exposition [36, Appendix B]), because $B$ is primitive with positive trace, there is a positive matrix $B_1$ SSE over $\mathbb{R}$ (in fact over $\mathbb{R}_+$) to $B$. And then, by arguments in [83], for some $m$ there are $m \times m$ matrices $A_2, B_2$ (obtained through row splittings of $A$ and $B_1$ ), with $B_2$ positive, such that $A$ is SSE over $\mathcal{R}$ to $A_2$; $B_1$ is SSE over $\mathbb{R}$ (in fact over $\mathbb{R}_+$) to a positive matrix $B_2$; and there is a matrix $U$ in $\mathrm{SL}(m, \mathbb{R})$ such that $U^{-1}A_2U = B_2$. Because $\mathrm{SL}(m, \mathcal{R})$ is dense in $\mathrm{SL}(m, \mathbb{R})$, and $B_2$ is positive, there is a $V$ in $\mathrm{SL}(m, \mathcal{R})$ such that $V^{-1}A_2V$ is positive. This matrix $(V^{-1}A_2)(V)$ is SSE over $\mathcal{R}$ to the matrix $(V)(V^{-1}A_2) = A$. □

After more than 20 years, the GSC remains open even in the case $\mathcal{R} = \mathbb{R}$. Still, the GSC seems correct. What we lack is a proof.

# Chapter 5: Explicit examples in $NK_1(\mathcal{R})$

## 5.1 Introduction

The purpose of this note is simply to exhibit explicit matrices representing nonzero classes in the algebraic $K$-theory group $NK_1(\mathcal{R})$ (and thereby in $Nil_0(\mathcal{R})$), for some rings $\mathcal{R}$ for which $NK_1(\mathcal{R})$ arises as an obstruction in [33] and [35]. In [33], $\mathcal{R}$ is the integral group ring of a finite group $G$; our example is for $G = \mathbb{Z}/4\mathbb{Z}$. (Proof of the nontriviality of $NK_1(\mathbb{Z}[\mathbb{Z}/4])$ can be found in [22] or [73].) In [35], $\mathcal{R}$ is a subring of $\mathbb{R}$; our example is for $\mathcal{R} = \mathbb{Q}[t^2, t^3, z, z^{-1}]$ (which has many embeddings into $\mathbb{R}$).

It seems to be difficult to locate explicit examples of this sort in the literature. The arguments to follow give some indication as to why that might be the case. The arguments are elementary, and only require carefully tracing through standard arguments and constructions in algebraic K-theory. However, the actual computation becomes lengthy, and leads to fairly large matrix examples.

The computation might be of interest for someone new to $K$-theory, as an example of the complication buried in certain exact sequence arguments.

## 5.2 Setup

We will require little setup, all of which can be found in [19], or [18]. Always, $\mathcal{R}$ is an associative ring with 1. We consider $K_1(\mathcal{R})$ as the group $GL(\mathcal{R})/El(\mathcal{R})$, where $GL(\mathcal{R})$ denotes the stabilized general linear group over $\mathcal{R}$, and $El(\mathcal{R})$ the stabilized elementary subgroup of $GL(\mathcal{R})$. There is a map $K_1(\mathcal{R}[t]) \to K_1(\mathcal{R})$ induced by $t \mapsto 0$, and the kernel of this map is defined to be $NK_1(\mathcal{R})$. Higman's trick implies $NK_1(\mathcal{R})$ is the set of elements of $K_1(\mathcal{R}[t])$ which contain a matrix of the form $I - tN$, with $N$ a nilpotent matrix over $\mathcal{R}$.

The group $Nil_0(\mathcal{R})$ is defined from $\mathbf{Nil}\mathcal{R}$, the nilpotent category over $\mathcal{R}$. The objects of this category are pairs $(P, f)$, where $P$ is a finitely generated projective $\mathcal{R}$-module and $f$ is a nilpotent endomorphism of $P$. A morphism $(P, f) \to (Q, g)$ is an $\mathcal{R}$ module homomorphism $h : P \to Q$ such that $hf = gh$. $\mathbf{Nil}\mathcal{R}$ acquires an exact structure via the forgetful functor $\mathbf{Nil}\mathcal{R} \to \mathbf{Proj}\mathcal{R}$ given by $(P, f) \mapsto P$: a sequence in $\mathbf{Nil}\mathcal{R}$ is exact if its image under this forgetful functor is exact. One may then consider the $K$-group $K_0(\mathbf{Nil}\mathcal{R})$ of the exact category $\mathbf{Nil}\mathcal{R}$ (see [19, II.7]). The cokernel of the map $K_0(\mathbf{Proj}\mathcal{R}) \to K_0(\mathbf{Nil}\mathcal{R})$ given by $[P] \mapsto [(P, 0)]$ is denoted $Nil_0(\mathcal{R})$. A well-known isomorphism $NK_1(\mathcal{R}) \to Nil_0(\mathcal{R})$ is induced by $N \mapsto I - tN$, where $N$ denotes a nilpotent matrix over $\mathcal{R}$ (viewed as an endomorphism of $\mathcal{R}^n$, where $N$ is $n \times n$).

For $k \geq 1$, we recall an important endomorphism of $NK_1(\mathcal{R})$. The Verschiebung map $V_k$ acts on $NK_1(\mathcal{R})$ via $V_k([1 - tN]) = [1 - t^k N]$, and acts on

$Nil_0(\mathcal{R})$ via

$$V_k([N]) = \left[ \begin{pmatrix} 0 & 0 & \cdots & 0 & N \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \right].$$

For $k \geq 1$ there are associated endomorphisms called the Frobenius maps $F_k$, which act on $NK_1(\mathcal{R})$ via $F_k([I - tN]) = [I - tN^k]$, and acts on $Nil_0(\mathcal{R})$ via $F_k([N]) = [N^k]$. We will only require the Verschiebung map in the present note.

### 5.3 A matrix nontrivial in $NK_1(\mathbb{Q}[t^2, t^3, z, z^{-1}])$

Consider the ring $\mathbb{Q}[t^2, t^3, z, z^{-1}]$. For consistency, we let $NK_1(\mathbb{Q}[t^2, t^3, z, z^{-1}])$ denote the kernel of the map $K_1(\mathbb{Q}[t^2, t^3, z, z^{-1}, s]) \stackrel{s \mapsto 0}{\to} K_1(\mathbb{Q}[t^2, t^3, z, z^{-1}])$. In this section, we show the following.

**Theorem 5.1.** *1. The class of the matrix*

$$\begin{pmatrix} 1 - (1 + z^{-1})s^4t^4 & (z-1)(s^2t^2 - s^3t^3) \\ (1 - z^{-1})(s^2t^2)(1 + st + s^2t^2 + s^3t^3) & 1 + (z-1)(s^4t^4) \end{pmatrix}$$

*is not zero in $NK_1(\mathbb{Q}[t^2, t^3, z, z^{-1}])$.*

*2. The class of the matrix*

$$\begin{pmatrix} 0 & 0 & 0 & (1-z)t^2 & 0 & (1-z)(-t^3) & (1-z^{-1})t^4 & 0 & 0 & 0 \\ 0 & 0 & (z^{-1}-1)t^2 & 0 & (z^{-1}-1)t^3 & 0 & (z^{-1}-1)t^4 & (1-z)t^4 & (z^{-1}-1)t^5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

*is non-zero in* $Nil_0(\mathbb{Q}[t^2, t^3, z, z^{-1}]$.

It follows from [45, Theorem 5.4] that the nilpotent matrix in (2) of 5.1 is not strong shift equivalent over the ring to the zero matrix (see Section 5). Since this ring may be embedded in $\mathbb{R}$, this provides an example of a matrix over a subring of the reals which is shift equivalent, but not strong shift equivalent, to zero.

The remainder of this section is devoted to proving 5.1. An outline for the construction is as follows: we begin with a non-zero class in $K_1(\mathbb{Q}[t, s]/I)$ which lies in the kernel of the map induced by $s \mapsto 0$. Such classes are easy to find. We then proceed by applying a collection of maps to this element, taking care that at each stage of the composition, the element remains non-zero, and ending in $K_1(\mathbb{Q}[t^2, t^3, z, z^{-1}, s])$. The resulting element still lies in the kernel upon sending $s \mapsto 0$, so the final element lies in $NK_1[\mathbb{Q}[t^2, t^3, z, z^{-1}])$. The maps which will be applied are shown in the diagram below, starting bottom left and ending top right:

$$\begin{tikzcd}
 & K_0(\mathbb{Q}[t^2,t^3,s],I) \arrow[r, "(p_2)_*"] & K_0(\mathbb{Q}[t^2,t^3,s]) \arrow[r, "\cdot z"] & K_1(\mathbb{Q}[t^2,t^3,z,z^{-1},s]) \\
K_1(\mathbb{Q}[t,s]/I) \arrow[r, "\partial"] & K_0(\mathbb{Q}[t,s],I) \arrow[u, "\phi_*", "\cong"'] & &
\end{tikzcd}$$

Consider the ideal $I = t^2\mathbb{Q}[t] \subset \mathbb{Q}[t,s]$. There is an exact sequence

$$0 \to I \to \mathbb{Q}[t,s] \to \mathbb{Q}[t,s]/I \to 0$$

which yields an exact sequence in $K$-theory (see [18, 2.5.4])

$$\cdots \to K_1(\mathbb{Q}[t,s]) \xrightarrow{\pi_*} K_1(\mathbb{Q}[t,s]/I) \xrightarrow{\partial} K_0(\mathbb{Q}[t,s],I) \xrightarrow{(p_2)_*} K_0(\mathbb{Q}[t,s]) \xrightarrow{\pi_*} K_0(\mathbb{Q}[t,s]/I)$$

where $K_i(\mathbb{Q}[t,s],I)$ refers to the relative groups defined via the double

$$D(\mathbb{Q}[t,s],I) = \{(x,y) \in \mathbb{Q}[t,s] \times \mathbb{Q}[t,s] | x-y \in I\}$$

of $\mathbb{Q}[t,s]$ along $I$ (see [18, 1.5.3]), $\pi_*$ is induced by $\pi : \mathbb{Q}[t,s] \to \mathbb{Q}[t,s]/I$, $(p_2)_*$ is induced by the projection onto the second coordinate $p_2 : D(\mathbb{Q}[t,s],I) \to \mathbb{Q}[t,s]$, and $\partial$ is the boundary map. Consider the class $1+ts \in K_1(\mathbb{Q}[t,s]/I)$.

**Step 1 - computing $\partial$:** Since $1+ts$ is not in the image of $\pi_* : K_1(\mathbb{Q}[t,s]) \to K_1(\mathbb{Q}[t,s]/I)$, $\partial(1+ts)$ represents a non-zero class in $K_0(\mathbb{Q}[t,s],I)$. We proceed by computing $\partial(1+ts)$ explicitly, which is done via a standard clutching construction. An outline of this can be found in [18, 2.5.4].

First consider $M_{1+ts} = \{(x,y) \in \mathbb{Q}[t,s]^2 | y - x(1+ts) \in I\}$ (thinking of $x,y$ as row vectors). This is a projective $D(\mathbb{Q}[t,s],I)$-module, and we get $\partial(1+ts) =$

$[M_{1+ts}] - [D(\mathbb{Q}[t,s], I)]$. For computational purposes it turns out to be more useful to compute the class of idempotent matrices representing $[M_{1+ts}]$ and $[D(\mathbb{Q}[t,s], I)]$. For this, first note the product

$$\begin{aligned} A &= \begin{pmatrix} 1+st+s^2t^2+s^3t^3 & -s^2t^2 \\ s^2t^2 & 1-st \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1+st \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -(1+st) & 1 \end{pmatrix} \begin{pmatrix} 1 & 1+st \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

is a lift of $\begin{pmatrix} 1+st & 0 \\ 0 & 1-st \end{pmatrix}$ (so $\pi(A) = \begin{pmatrix} 1+st & 0 \\ 0 & 1-st \end{pmatrix}$), and there is an isomorphism

$$j : M_{1+st} \oplus M_{1-st} \to D(\mathbb{Q}[t,s], I)^2$$

given by

$$\begin{aligned} j : \begin{pmatrix} (x,y) & (u,v) \end{pmatrix} &\to (\begin{pmatrix} x & u \end{pmatrix} A, \begin{pmatrix} y & v \end{pmatrix}) \\ &\to \begin{pmatrix} (\pi_1(\begin{pmatrix} x & u \end{pmatrix} A), \pi_1(\begin{pmatrix} y & v \end{pmatrix})) & (\pi_2(\begin{pmatrix} x & u \end{pmatrix} A), \pi_2(\begin{pmatrix} y & v \end{pmatrix})) \end{pmatrix} \\ &= \begin{pmatrix} (\pi_1(\begin{pmatrix} x & u \end{pmatrix} A), y) & (\pi_2(\begin{pmatrix} x & u \end{pmatrix} A), v) \end{pmatrix} \end{aligned}$$

where $\pi_1$ and $\pi_2$ denote projection on to the 1st and 2nd component, respectively. Using this isomorphism, one can check that

$$\begin{tikzcd} M_{1+st} \oplus M_{1-st} \arrow[r, "j"] \arrow[d, "id \oplus 0"] & D(\mathbb{Q}[t,s], I)^2 \arrow[d, "B"] \\ M_{1+st} \oplus M_{1-st} \arrow[r, "j"] & D(\mathbb{Q}[t,s], I)^2 \end{tikzcd}$$

commutes, where $B = \begin{pmatrix} (1-s^4t^4, 1) & ((-s^2t^2)(1+st+s^2t^2+s^3t^3), 0) \\ (s^3t^3 - s^2t^2, 0) & (s^4t^4, 0) \end{pmatrix}$. Thus, the idempotent matrix $B$ represents the class of $M_{1+ts}$ in $K_0(\mathbb{Q}[t,s], I)$. For ease of notation in what follows, we will express $B$ instead as an ordered pair of matrices $B = (B_1, B_2)$ coming from each of the components of the entries of $B$, so

$$B = (\begin{pmatrix} 1-s^4t^4 & (-s^2t^2)(1+st+s^2t^2+s^3t^3) \\ s^3t^3 - s^2t^2 & s^4t^4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})$$

and

$$B_1 = \begin{pmatrix} 1-s^4t^4 & (-s^2t^2)(1+st+s^2t^2+s^3t^3) \\ s^3t^3 - s^2t^2 & s^4t^4 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Since $[D(\mathbb{Q}[t,s], I)] = [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]$ in idempotent form, we get $\partial(1+ts) = [B] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]$. For notational ease, let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

**Step 2 - Excision isomorphism $\phi_*$:** Since $K_0$ has the excision property, there are isomorphisms $K_0(\mathbb{Q}[t,s], I) \overset{\gamma_{1,*}^{-1}}{\cong} K_0(I) \overset{\gamma_{2,*}}{\cong} K_0(\mathbb{Q}[t^2,t^3,s], I)$. Let

$$\phi_* = \gamma_{2,*} \circ \gamma_{1,*}^{-1} : K_0(\mathbb{Q}[t,s], I) \to K_0(\mathbb{Q}[t^2,t^3,s], I)$$

denote the composition of these isomorphisms. Here $K_0(I)$ denotes $K_0$ of the non-unital ring $I$ [18, 1.5.7]. This is defined by unitizing $I$, i.e. forming the ring $I_+ = I \oplus \mathbb{Z}$ with multiplication given by $(x,n) \cdot (y,m) = (xy + ny + mx, mn)$, and defining $K_0(I)$ to be the kernel of the induced map from the surjection on to the second factor, $K_0(I) = ker(K_0(I_+) \to K_0(\mathbb{Z}))$. The isomorphism $\gamma_{1,*} : K_0(I) \to$

$K_0(\mathbb{Q}[t,s],I)$ is induced by the map $(x,n)\to(n,n+x)$. Letting $e_2=(A^T)^{-1}DA^T=\begin{pmatrix} 1-s^4t^4 & s^2t^2-s^3t^3 \\ s^2t^2(1+st+s^2t^3+s^3t^3) & s^4t^4 \end{pmatrix}$ (transposes appear because in the isomorphism $j$ above we are acting on row vectors), a computation (see [18, 1.5.9] for details) gives

$$\gamma_*^{-1}([B]-[P,P])=[e_2-P,P]-[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},P]$$

Applying $\gamma_{2,*}$ gives

$$\gamma_{2,*}([e_2-P,P]-[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},P])=[P,e_2]-[P,P]$$

so

$$\phi_*([B]-[P,P])=[P,e_2]-[P,P]\in K_0(\mathbb{Q}[t^2,t^3,s],I)$$

**Step 3 - computing $(p_2)_*$:** The map $(p_2)_*:K_0(\mathbb{Q}[t^2,t^3,s],I)\to K_0(\mathbb{Q}[t^2,t^3,s])$ is induced by the projection $p_2:D(\mathbb{Q}[t^2,t^3,s],I)\to\mathbb{Q}[t^2,t^3,s]$ onto the second coordinate. Thus $(p_2)_*([P,e_2]-[P,P])=[e_2]-[P]$. We claim this class is non-zero in $K_0(\mathbb{Q}[t^2,t^3,s])$. To see this, note there is a splitting map $\varphi:\mathbb{Q}[s]\to\mathbb{Q}[t^2,t^3,s]$ for $q:\mathbb{Q}[t^2,t^3,s]\to\mathbb{Q}[t^2,t^3,s]/I=\mathbb{Q}[s]$, which implies that the map $q_*:K_1(\mathbb{Q}[t^2,t^3,s])\to K_1(\mathbb{Q}[t^2,t^3,s]/I)$ is surjective. This in turn implies that the boundary map $\partial$ in the exact sequence

$$\cdots\to K_1(\mathbb{Q}[t^2,t^3,s])\xrightarrow{q_*}K_1(\mathbb{Q}[t^2,t^3,s]/I)\xrightarrow{\partial}K_0(\mathbb{Q}[t^2,t^3,s],I)\xrightarrow{(p_2)_*}K_0(\mathbb{Q}[t^2,t^3,s])\to\cdots$$

must be zero. Thus $(p_2)_*$ must be injective. Altogether we have the non-zero class $[e_2]-[P]\in K_0(\mathbb{Q}[t^2,t^3,s],I)$.

**Step 4 - computing $\cdot$ z:** Finally, for any ring $T$, there is an injective map (see [19, III.3.5.2]) $\cdot z : K_0(T) \to K_1(T[z, z^{-1}])$ given by $\cdot z : [Q] \to [I + (z-1)Q]$, where $Q$ is an idempotent matrix over $T$. Thus we apply this map to the idempotent $e_2$ to get $[I + (z-1)e_2] \in K_1(\mathbb{Q}[t^2, t^3, s, z, z^{-1}])$, and to $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ to get the difference

$$\begin{aligned}
&\cdot z([e_2] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]) \\
&= [I + (z-1)e_2] - [\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}] \\
&= [\begin{pmatrix} 1 + (z-1)(1 - s^4t^4) & (z-1)(s^2t^2 - s^3t^3) \\ (z-1)(s^2t^2)(1 + st + s^2t^2 + s^3t^3) & 1 + (z-1)(s^4t^4) \end{pmatrix}] - [\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}] \\
&= [\begin{pmatrix} z^{-1} + (1 - z^{-1})(1 - s^4t^4) & (z-1)(s^2t^2 - s^3t^3) \\ (1 - z^{-1})(s^2t^2)(1 + st + s^2t^2 + s^3t^3) & 1 + (z-1)(s^4t^4) \end{pmatrix}] \\
&= [\begin{pmatrix} 1 - (1 + z^{-1})s^4t^4 & (z-1)(s^2t^2 - s^3t^3) \\ (1 - z^{-1})(s^2t^2)(1 + st + s^2t^2 + s^3t^3) & 1 + (z-1)(s^4t^4) \end{pmatrix}]
\end{aligned}$$

in $K_1(\mathbb{Q}[t^2, t^3, s, z, z^{-1}])$. One can check easily that the above class maps to $[I]$ under the map $s \to 0$, and hence lies in $NK_1(\mathbb{Q}[t^2, t^3, z, z^{-1}, s]$. Lastly, non-triviality of the class was justified at each stage.

To find the corresponding class in $Nil_0$, let $I - M$ denote this matrix found above, so we have $M$ as

$$M = \begin{pmatrix} (1 - z^{-1})s^4t^4 & (1 - z)(s^2t^2 - s^3t^3) \\ (z^{-1} - 1)(s^2t^2)(1 + st + s^2t^2 + s^3t^3) & (1 - z)(s^4t^4) \end{pmatrix} = \sum_{i=1}^{5} s^i M_i$$

with the $M_i$ over $\mathbb{Q}[t^2, t^3, z, z^{-1}, s]$. Under the isomorphism $NK_1 \to Nil_0$ we obtain (see [45]) a nilpotent matrix $N$ over $\mathbb{Q}[t^2, t^3, z, z^{-1}, s]$,

$$N = \begin{pmatrix} M_1 & M_2 & M_3 & M_4 & M_5 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & (1-z)t^2 & 0 & (1-z)(-t^3) & (1-z^{-1})t^4 & 0 & 0 & 0 \\ 0 & 0 & (z^{-1}-1)t^2 & 0 & (z^{-1}-1)t^3 & 0 & (z^{-1}-1)t^4 & (1-z)t^4 & (z^{-1}-1)t^5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

which is nontrivial as an element of $Nil_0(\mathbb{Q}[t^2, t^3, z, z^{-1}, s])$.

One can of course use the above technique to generate many more explicit non-zero classes: simply start with any unit of the form $a + bst$ in $\mathbb{Q}[t, s]/t^2\mathbb{Q}[t, s]$, $a, b \in \mathbb{Q}$, and apply the sequence of maps $\cdot z(p_2)_*\phi_*\partial$ to $a + bst$ as above.

## 5.4 A matrix nontrivial in $\mathbb{Z}G$, for $G = \mathbb{Z}/4$

This section is concerned with the integral group rings of finite cyclic groups. Let $p$ be a prime, and $\mathbb{Z}/p^n$ denote a cyclic group of order $p^n$. In [72, Theorem 3.12] (except for a few cases), and later in [73], it is shown that $NK_1(\mathbb{Z}[\mathbb{Z}/p^n]) \neq 0$ for $n \geq 2$. In [22] it is also shown that $NK_1(\mathbb{Z}[G])$ is not zero for $G = \mathbb{Z}/4$, along with $G = D_4$, the dihedral group. The technique in [73] is an extension of that found in [22, Theorem1.4], and uses the Milnor square

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{Z}/p^n] \arrow[r] \arrow[d] & \mathbb{Z}[\zeta_{p^n}] \arrow[d] \\
\mathbb{Z}[\mathbb{Z}/p] \arrow[r] & \mathbb{Z}[\zeta_{p^n}]/(1-\zeta_{p^n}^p)
\end{tikzcd}$$

where $\zeta_n$ denote a primitive $n$th root of unity, and $\mathbb{Z}[\zeta_n]$ the ring of integers of $\mathbb{Q}[\zeta_n]$. The bottom right term is isomorphic to $\mathbb{Z}_p[t]/(t^p)$ ( [73]), and the square yields the Mayer-Vietoris sequence

$$\begin{aligned}
&NK_2(\mathbb{Z}[\mathbb{Z}/p^n]) \to NK_2(\mathbb{Z}[\zeta_{p^n}]) \oplus NK_2(\mathbb{Z}[\mathbb{Z}/p]) \to NK_2(\mathbb{Z}_p[t]/(t^p)) \\
&\to NK_1(\mathbb{Z}[\mathbb{Z}/p^n]) \to NK_1(\mathbb{Z}[\zeta_{p^n}]) \oplus NK_1(\mathbb{Z}[\mathbb{Z}/p]) \to NK_1(\mathbb{Z}_p[t]/(t^p)) \to \cdots
\end{aligned} \tag{5.1}$$

Since $\mathbb{Z}[\zeta_{p^n}]$ is regular, $NK_i(\mathbb{Z}[\zeta_{p^n}]) = 0$ for $i = 1, 2$, and $NK_1(\mathbb{Z}[\mathbb{Z}/p]) = 0$ from [21]. Thus $NK_1(\mathbb{Z}[\mathbb{Z}/p^n])$ is isomorphic to the cokernel of $NK_2(\mathbb{Z}[\mathbb{Z}/p]) \to NK_2(\mathbb{Z}_p[t]/(t^p))$. A presentation of this cokernel is given in [73] using the computation of $NK_2(\mathbb{Z}_p[t]/(t^p))$ by van der Kallen and Stienstra found in [84].

We use this method to produce concrete non-zero classes in $NK_1(\mathbb{Z}[\mathbb{Z}/4])$.

For the case at hand, namely $\mathbb{Z}[\mathbb{Z}/4]$, a $p=2$ case of the argument also appears in Weibel [22, Theorem 1.4], and we mimic the notation found there.

**Theorem 5.2.** *The class of the matrix* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *in* $NK_1(\mathbb{Z}[\mathbb{Z}/4])$ *with*

$$A = 1 - (1-\sigma^2)(x - 2x^2 + 2x^3 - \sigma + x\sigma + x^2\sigma)$$

$$B = (\sigma^2 - 1)(1 + 2x - x^2 - x^3 - 2x^4 + \sigma - x\sigma - 2x^2\sigma - 3x^3\sigma + 2x^4\sigma)$$

$$C = (\sigma^2 - 1)(-1 + 2x - 5x^2 + 7x^3 - 3x^4 + 2x^5 - \sigma + 2x\sigma - 2x^3\sigma + 3x^4\sigma - 2x^5\sigma)$$

$$D = 1 - (1-\sigma^2)(2 + x - 2x^2 - 4x^4 - 2x^5 + \sigma - 3x\sigma - x^2\sigma - 4x^3\sigma + 6x^4\sigma - 4x^5\sigma + 4x^6\sigma)$$

*is non-zero.*

To construct a corresponding non-zero class in $Nil_0(\mathbb{Z}[\mathbb{Z}/4])$, one could now apply Higman's trick. Since the matrix contains powers of $x$ up to and including 6, this would yield a $12 \times 12$ nilpotent matrix.

The remainder of the section is devoted to verifying 5.2.

Let $G = [\mathbb{Z}/4]$ be the cyclic group of order 4 generated by $\sigma$, and let $\mathcal{R} = \mathbb{Z}[G]$. As described above we have the square

$$\begin{tikzcd} \mathcal{R} \arrow[r, "\sigma \to i"] \arrow[d, "\sigma^2 \to 1"'] & \mathbb{Z}[i] \arrow[d, "i \to 1+\epsilon"] \\ \mathbb{Z}[\mathbb{Z}/2] \arrow[r, "q"] & \mathbb{F}_2[\epsilon]/(\epsilon^2) \end{tikzcd}$$

with $q$ the map induced by the quotient $\mathbb{Z} \to \mathbb{F}_2$, yielding the Mayer-Vietoris sequence (see [22]), a part of which reads

$$\cdots \to NK_2(\mathbb{Z}[\mathbb{Z}/2]) \xrightarrow{q} NK_2(\mathbb{F}_2[\epsilon]/(\epsilon^2)) \xrightarrow{\partial} NK_1(\mathcal{R}) \to \cdots$$

Furthermore, Lemma 1.2 in [22] implies $Im(q) = ker(D : NK_2(\mathbb{F}_2[\epsilon]/(\epsilon^2)) \to \Omega_{\mathbb{F}_2[x]}$, where $\Omega_{\mathbb{F}_2[x]}$ denotes the Kähler differentials of $\mathbb{F}_2[x]$, and $D$ is the map $D(\langle f\epsilon, g + g'\epsilon\rangle) = f\,dg$. Here $\langle\,,\rangle$ denotes the Dennis-Stein symbol in $K_2$ (see [19, III.5.11]). Thus, choosing for example $\langle\epsilon, x+\epsilon\rangle$, we have $D(\langle\epsilon, x+\epsilon\rangle) = dx \neq 0$, so $\langle\epsilon, x+\epsilon\rangle \notin Im(q)$. It follows that $\partial(\langle\epsilon, x+\epsilon\rangle) \neq 0$ in $NK_1(\mathcal{R})$.

It remains to compute the boundary map $\partial(\langle\epsilon, x+\epsilon\rangle)$. This is obtained (see [19, III.5.8]) from the composition (bottom left to top right)

$$\begin{tikzcd}
 & K_1(\mathcal{R}[x], (1-\sigma^2)) \arrow[r, "j"] \arrow[d, "\psi", "\cong"'] & K_1(\mathcal{R}[x]) \\
K_2(\mathbb{F}_2[\epsilon, x]/(\epsilon^2)) \arrow[r, "\partial_1"] & K_1(\mathbb{Z}[i][x], (2)) &
\end{tikzcd}$$

Here $\psi$ is the induced map from $\mathcal{R}[x] \to \mathbb{Z}[i][x]$, which is an isomorphism, since the map $\sigma \to i$ takes the ideal $(1-\sigma^2)$ isomorphically onto the ideal $(2)$. The map $\partial_1$ is the standard boundary map for the long exact from an ideal, whose computation is routine[1], yielding

$$\partial_1(\langle\epsilon, x+\epsilon\rangle) =$$

$$[YZ]$$

[1]We include the calculation in the last section, for completeness.

where

$$Y := e_{21}(-x+1-i+(1-i)x^2)e_{12}(1-i)e_{21}(x+i-1)e_{12}(i-1)$$

$$Z := e_{12}(1)e_{21}(-1)e_{12}(1)e_{12}((i-1)x-1)e_{21}(1+(i-1)x)e_{12}((i-1)x-1)$$

Lifting this up to $K_1(\mathcal{R}[x],(1-\sigma^2))$ via the vertical isomorphism $\psi$ gives

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with

$$A = 1-(1-\sigma^2)(x-2x^2+2x^3-\sigma+x\sigma+x^2\sigma)$$

$$B = (\sigma^2-1)(1+2x-x^2-x^3-2x^4+\sigma-x\sigma-2x^2\sigma-3x^3\sigma+2x^4\sigma)$$

$$C = (\sigma^2-1)(-1+2x-5x^2+7x^3-3x^4+2x^5-\sigma+2x\sigma-2x^3\sigma+3x^4\sigma-2x^5\sigma)$$

$$D = 1-(1-\sigma^2)(2+x-2x^2-4x^4-2x^5+\sigma-3x\sigma-x^2\sigma-4x^3\sigma+6x^4\sigma-4x^5\sigma+4x^6\sigma)$$

Applying $j$ yields the class of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $K_1(\mathcal{R}[x])$. This class is our desired element in $NK_1(\mathcal{R})$.

The technique above can be repeated, using other symbols $\langle\ ,\ \rangle$ in $K_2(\mathbb{F}_2[\epsilon])$, although the computations become quite lengthy.

## 5.5 Strong shift equivalence

Let $A, B$ be square matrices over a ring $\mathcal{R}$ (not necessarily of the same size).

$A$ and $B$ are *elementary strong shift equivalent over* $\mathcal{R}$ (ESSE-$\mathcal{R}$) if there exist matrices $U, V$ over $\mathcal{R}$ such that $A = UV$ and $B = VU$. $A$ and $B$ are *strong shift equivalent over* $\mathcal{R}$ *(SSE-*$\mathcal{R}$*)* if there are matrices $A = A_0, A_1, \ldots, A_\ell = B$ such that for $1 \leq i \leq \ell$, $A_i$ and $A_{i-1}$ are ESSE-$\mathcal{R}$. $A$ and $B$ are *shift equivalent* over $\mathcal{R}$ if there exist matrices $U, V$ over $\mathcal{R}$ and $\ell$ in $\mathbb{N}$ such that $A^\ell = UV, B^\ell = VU, AU = UB$ and $VA = BV$.

It is proved in [45] that if $A$ is either an invertible matrix over $\mathcal{R}$ or a nilpotent matrix, and $N$ is any nilpotent matrix over $\mathcal{R}$, then the matrix $A \oplus N$ is SSE-$\mathcal{R}$ to $A$ if and only if $N$ is trivial as an element of $Nil_0(\mathcal{R})$. Moreover, if $B$ is SE-$\mathcal{R}$ to $A$, then there is a nilpotent $N$ such that $B$ is SSE-$\mathcal{R}$ to $A \oplus N$. (See [45] for further results, explanation and context.) A matrix is SE-$\mathcal{R}$ to $(0)$ if and only if it is nilpotent. So, in particular, $N$ is SSE-$\mathcal{R}$ to $(0)$ if and only if the nilpotent matrix $N$ is trivial as an element of $Nil_0(\mathcal{R})$.

## 5.6 Calculation of $\partial_1(\langle \epsilon, x + \epsilon \rangle)$

The last section contains the calculation of $\partial_1(\langle \epsilon, x + \epsilon \rangle)$. We let $I$ denote the ideal $(2)$ in $\mathbb{Z}[i]$, so that the map $\mathbb{Z}[i] \to \mathbb{F}_2[\epsilon]/(\epsilon^2)$ given by $i \mapsto 1 + \epsilon$ has kernel $I$. We also follow the notation in [19, III.5.11], so that, by definition,

$$\langle \epsilon, x + \epsilon \rangle = x_{ji}(-(x + \epsilon)(1 - \epsilon x)^{-1}) x_{ij}(-\epsilon) x_{ji}(x + \epsilon) x_{ij}((1 - \epsilon x)^{-1}\epsilon)(h_{ij}(1 - \epsilon x))^{-1}$$

where $i \neq j$, $x_{ij}, x_{ji}$ denote, as usual, generators in the Steinberg group, and for any unit $a$,

$$h_{ij}(a) = x_{ij}(a) x_{ji}(-a^{-1}) x_{ij}(a) x_{ij}(-1) x_{ji}(1) x_{ij}(-1)$$

Steinberg relations give reduce $\langle \epsilon, x+\epsilon \rangle$ to

$$X := x_{ji}(-x-\epsilon-\epsilon x^2)x_{ij}(-\epsilon)x_{ji}(x+\epsilon)x_{ij}(\epsilon)(h_{ij}(1-\epsilon x))^{-1}$$

Now $\partial_1$ is computed by composing up through the diagram, from bottom left to top right (see [19, III.5.71] - we have introduced the $\phi_i$ names for the maps),

$$\begin{tikzcd}
 & & GL(I) \arrow[r, "\phi_5"] \arrow[d, "\phi_4"] & K_1(\mathbb{Z}[i][x],(2)) \\
 & St(\mathbb{Z}[i][x]) \arrow[r, "\phi_3"] \arrow[d, "\phi_2"] & GL(\mathbb{Z}[i][x]) & \\
K_2(\mathbb{F}_2[\epsilon,x]/(\epsilon^2)) \arrow[r] & St(\mathbb{F}_2[\epsilon,x]/(\epsilon^2) & &
\end{tikzcd}$$

Now for simplicity we let $i=1, j=2$. We have $X \in St(\mathbb{F}_2[\epsilon,x]/(\epsilon^2)$, and lifting $X$ up using $\phi_2$ and applying $\phi_3$ gives $\phi_2(YZ) = X$ where

$$Y := e_{21}(-x+1-i+(1-i)x^2)e_{12}(1-i)e_{21}(x+i-1)e_{12}(i-1)$$

$$Z := e_{12}(1)e_{21}(-1)e_{12}(1)e_{12}((i-1)x-1)e_{21}(1+(i-1)x)e_{12}((i-1)x-1)$$

Lifting up via $\phi_4$ and applying $\phi_5$ gives the class of $YZ$ in $K_1(\mathbb{Z}[i][x],(2))$.

# Chapter 6: Isolating zero-dimensional dynamics on manifolds

## 6.1 Introduction

The notion of an isolated invariant set arises in a variety of situations, often in the realm of dynamical systems. While a general invariant set may be very complicated, in the case where the invariant set is isolated, many qualitative properties can be detected. For example, isolated sets and their isolating neighborhoods lie at the heart of the Conley index theory, a topic which has been extensively studied and employed (see [11] for a brief survey).

In the realm of hyperbolic dynamics, isolated sets are more often referred to as locally maximal sets, which play an important role in the theory [52, Section 6.4.d]. It is well known that in the hyperbolic setting (see for example [12, Section 18.4]), isolation is equivalent to having a local product structure, and any compact totally disconnected locally maximal set must be a shift of finite type. There are still open questions regarding isolation even for hyperbolic maps, see for instance [13]

Restrictions of Axiom A systems to basic sets form an essential class of examples. In this paper, we introduce "strongly isolated" systems, which can be viewed as a possible topological analogue to a basic set of an Axiom A map.

We consider for a homeomorphism $h$ of a manifold, what dynamics can occur as the restriction of $h$ to an isolated (or strongly isolated) zero-dimensional set.

We show that for a given compact zero-dimensional system $(X, f)$ and natural number $n \geq 3$, there exists a homeomorphism $g : M \to M$ of an $n$-dimensional manifold $M$ containing a strongly isolated invariant set on which $g$ is conjugate to $(X, f)$. We provide obstructions for a zero-dimensional system to occur as an isolated invariant set for a homeomorphism of a compact two-manifold. We also prove that any odometer which occurs as an invariant set in a 2-dimensional manifold must be the limit of periodic points.

## 6.2 Definitions and notation

For a compact metric space $X$ with a homeomorphism $f : X \to X$, a compact set $N$ is called an *isolating neighborhood* if the maximal invariant set $\mathrm{Inv}(N) = \bigcap_{n\in\mathbb{Z}} f^n(N)$ for $N$ is contained in $\mathrm{Int}(N)$. We call an invariant set $I \subset X$ *isolated* if there exists an isolating neighborhood $N$ of $I$ such that $I = \bigcap_{n\in\mathbb{Z}} f^n(U)$. In the case $I$ is a compact isolated invariant set, we will also refer to any neighborhood $U$ of $I$ for which $I = \bigcap_{n\in\mathbb{Z}} f^n(U)$ as an isolating neighborhood. Note that if $I$ is a compact isolated invariant set with isolating neighborhood $N$, then any neighborhood $U$ of $I$ such that $U \subset N$ is also an isolating neighborhood for $I$. Throughout, we will almost always consider the case where $X$ is a topological manifold.

We say $I$ is *strongly isolated* if it can be isolated with an isolating neighborhood $N$ such that $N \cap NW(f) \subset I$, where $NW(f)$ denote the set of nonwandering points of $X$ under $f$.

Both hyperbolic periodic orbits and attractors are examples of isolated sets, although the former may not be strongly isolated. The nonwandering set of an Axiom $A$ diffeomorphism is an important example of a strongly isolated set, and the notion of strongly isolated can be seen as one possible topological analogue of the notion of a basic set. As an example demonstrating the difference between isolation and strong isolation, one may consider the case of a subshift $(X,\sigma)$: in this case, a subshift $Y \subset X$ is isolated if and only if $Y$ is a subshift of finite type, and such $Y \subset X$ will never be strongly isolated.

For a homeomorphism $f : X \to X$, let us say the pair $(X, f)$ is *(orientably) isolatable in dimension* $n$ if $X$ can be embedded into a compact (orientable) $n$-manifold $M$ with an (orientation-preserving) homeomorphism $g : M \to M$ such that $g|_X = f$ and $X$ is isolated under $g$. We say the pair $(X, f)$ is *(orientably) strongly isolatable in dimension* $n$ if $X$ can be embedded into a compact (orientable) $n$-manifold $M$ with an (orientation-preserving) homeomorphism $g : M \to M$ such that $g|_X = f$ and $X$ is strongly isolated under $g$. Note that for a system $(X, f)$, being isolatable in dimension $n$ depends only on the topological conjugacy class of the system $(X, f)$.

For a homeomorphism $f : X \to X$ of a compact metric space $X$, the suspension space $\Sigma_f X$ (sometimes called mapping torus) is defined to be the quotient of $X \times \mathbb{R}$ under the relation $(x, t) \sim (f(x), t - 1)$. The suspension $\Sigma_f X$ comes equipped with a natural flow $\phi : \Sigma_f X \times \mathbb{R} \to \Sigma_f X$. In the case $X$ is a compact zero-dimensional metric space, $\Sigma_f X$ is a compact metric space locally homeomorphic to the product of a zero-dimensional space and an arc. We will assume throughout that our spaces are metrizable, and note that for a locally compact Hausdorff space $Y$, $Y$ is zero-dimensional if and only if $Y$ is totally disconnected. We use the term *compact zero-dimensional system* to refer to a pair $(X, f)$ where $f : X \to X$ is a homeomorphism and $X$ is a compact zero-dimensional space.

Unless otherwise stated, for a space $X$, $\check{H}^*(X)$ denotes Čech cohomology taken with integer coefficients.

## 6.3 Isolation in dimension three and higher

The goal of this section is to prove that any compact zero-dimensional system can be strongly isolated in a compact manifold of dimension 3 and higher. This relies on a embedding result of Moise, along with a technique to convert a compact invariant set in dimension $n$ into a compact strongly isolated invariant set in dimension $n+1$.

For the following lemma it will be convenient to identify $S^1$ as the interval $[-1, 1]$ with the endpoints identified.

**Lemma 6.1.** Let $f : M \to M$ be a homeomorphism of a manifold $M$, and suppose $X$ is a compact invariant set of $f$. There exists a homeomorphism $h : M \times S^1 \to M \times S^1$ such that $h|_{X\times\{0\}} = f$, and $X \times \{0\}$ is strongly isolated by $h$.

*Proof.* Let $\pi_2 : M \times [-1, 1] \to [-1, 1]$ be projection on to the second coordinate. There exists a continuous map $r : M \times [-1, 1] \to M \times [-1, 1]$ such that for all

$x \in M$ we have $r(x,-1) = (x,-1)$, $r(x,1) = (x,1)$, $\pi_2(r(x,t)) \geq t$, and for which $\pi_2(r(x,0)) = 0$ if and only if $x \in X$ (such a map is not hard to construct). The map $r$ gives a map $R : M \times S^1 \to M \times S^1$ in the obvious way. Let $F : M \times S^1 \to M \times S^1$ by $F(x,y) = (f(x), y)$. Then the composition $h = R \circ F$ is a homeomorphism of $M \times S^1$ for which $X \times \{0\} \subset M \times S^1$ is invariant and strongly isolated by $h$, and $h|_{X \times \{0\}} = f$.

□

*Remark* 6.2. In the case $(M, f)$ is $C^r$ with $r \in [1, \infty)$, the map $h$ in the Lemma can be made $C^r$. Indeed, the map $r$ in the proof can be constructed to be $C^\infty$, using for example a smooth version of Urysohn's Lemma.

*Remark* 6.3. The isolated dynamics of $(X \times \{0\}, h|_{X \times \{0\}})$ produced in the construction do not depend stably on $h$, as a perturbation of $h$ can leave all points forwardly asymptotic to $X \times \{1\}$.

Part (1) of the following result, due to Moise, follows immediately from [85, Ch. 13, Theorem 7]. Part (2) follows easily from (1). Let $\mathbb{D}$ denote the closed unit disk in $\mathbb{R}^2$.

**Theorem 6.1.** *[85, Ch. 13, Theorem 7] Let $f : X \to X$ be a homeomorphism of a compact totally disconnected space $X$.*

1. *There exists an embedding $i : X \hookrightarrow \mathbb{D}$ and homeomorphism $g : \mathbb{D} \to \mathbb{D}$ such that $g|_{i(X)} = f$ and $g|_{\partial \mathbb{D}} = id$.*

2. *For any two-manifold $M$ there exists an embedding $i : X \hookrightarrow M$ and homeomorphism $g : M \to M$ such that $g|_{i(X)} = f$.*

Theorem 6.1 along with Lemma 1 gives our main realization result.

**Theorem 6.2.** *Let $M$ be a compact two-manifold and let $f : X \to X$ be a homeomorphism of a compact zero-dimensional space $X$. Then $(X, f)$ can be realized as a strongly isolated invariant set for a homeomorphism of $M \times \mathbb{T}^n$ for $n \geq 1$.*

*Proof.* This follows immediately for $M \times \mathbb{T}$ from Theorem 6.1 and Lemma 6.1. Iterating the construction then gives the result for $M \times \mathbb{T}^n$ for all $n$. □

## 6.4 Examples in dimension two

The question of isolating zero-dimensional systems in dimension two is significantly different from dimension three and above, where the realization result from Theorem 6.2 can be used. The next few sections present a variety of examples of zero-dimensional systems which are isolatable in dimension 2. The list is not intended to be exhaustive with regard to the techniques used, but instead to demonstrate a wide range of systems using different constructions.

We will occasionally use the following result in the examples below. We record it as a lemma.

**Lemma 6.4.** Let $f : X \to X$, $g : Y \to Y$ be continuous maps with $X, Y$ compact, and suppose $\pi : X \to Y$ is a surjective continuous map such that $\pi f = g\pi$.

1. If $f$ and $g$ are homeomorphisms for which $A \subset X$ is a compact isolated

invariant set under $f$ such that $\pi^{-1}(\pi(A)) = A$, then $\pi(A)$ is an isolated invariant set under $g$.

2. Suppose $A \subset X$ is a compact invariant set under $f$ for which there exists a neighborhood $U$ of $A$ such that $\bigcap\limits_{n \in \mathbb{N}} f^{-n}(U) = A$, and $\pi^{-1}(\pi(A)) = A$. Then there exists an open neighborhood $V$ of $\pi(A)$ such that $\bigcap\limits_{n \in \mathbb{N}} g^{-n}(V) = A$

*Proof.* For part (1), suppose $U$ is an isolating neighborhood for $A$ under $f$. Then the fact that $\pi^{-1}(\pi(A)) = A$ along with compactness implies there exists an open neighborhood $V$ of $\pi(A)$ such that $\pi^{-1}(V) \subset U$. Since $U$ is an isolating neighborhood for $A$ under $f$, it follows that $\bigcap\limits_{n \in \mathbb{Z}} f^n(\pi^{-1}(V)) = A$. Thus $\bigcap\limits_{n \in \mathbb{Z}} \pi^{-1}(g^n(V)) = A$, so $\pi^{-1}(\bigcap\limits_{n \in \mathbb{Z}} g^n(V)) = A$. Hence we have $\bigcap\limits_{n \in \mathbb{Z}} g^n(V) \subset \pi(A)$; but clearly $\pi(A) \subset \bigcap\limits_{n \in \mathbb{Z}} g^n(V)$ as well, so $A = \bigcap\limits_{n \in \mathbb{Z}} g^n(V)$.

The proof of part (2) is analogous to that of part (1), with the observation that for any $V \subset Y$ and $n \in \mathbb{N}$ one has $f^{-n}\pi^{-1}(V) = \pi^{-1}g^{-n}(V)$. $\square$

## 6.5 Isolating shifts of finite type in dimension two

The collection of standard Smale horseshoe maps ( [52, Section 2.5.c]) on $S^2$ allow one to realize any subshift of finite type as an isolated subsystem in dimension two. However, the question of which shifts of finite type can occur as a strongly isolated subsystem is more diffcult, the set of shifts of finite type which can occur in Axiom A systems has been investigated, especially in the context of Smale diffeomorphisms of surfaces. For example, it is known that there are infinitely many shifts of finite that can not occur as a basic set of a Smale diffeomorphism of a surface [86]. A characterization of the set of polynomials over $\mathbb{Z}/2[t]$ which can occur as the reduced zeta function of a Smale diffeomorphism of a surface has been obtained by Fried in [87]. More generally, restrictions are known to exist on zeta functions of an isolated system, under certain conditions for $C^1$ diffeomorphisms [88] and even homeomorphisms [89].

## 6.6 Isolating (Denjoy) minimal shifts in dimension two

This example exhibits very different behavior than the standard shift of finite type examples coming from horseshoes. Let $f : S^1 \to S^1$ be a Denjoy homeomorphism (see [52, 12.2]), and let $\Sigma \subset S^1$ be the unique minimal Cantor system invariant under $f$. Then the realization result, Theorem 6.2, applied to $f : S^1 \to S^1$ yields a homeomorphism $\tilde{f} : \mathbb{T}^2 \to \mathbb{T}^2$ containing $(\Sigma, f|_\Sigma)$ as a strongly isolated invariant system. Note the system $(\Sigma, f|_\Sigma)$ is minimal and expansive.

## 6.7 Isolating a mixing strictly sofic shift in dimension two

This example produces a homeomorphism $F$ of a compact two-manifold $M$ satisfying the following:

1. $M$ is homeomorphic to an annulus
2. $F$ is the identity on $\partial M$
3. $M$ contains a compact invariant set $Z$ on which $F$ is topologically conjugate

to a mixing strictly sofic shift

4. $F$ can be constructed to be either orientation preserving or orientation reversing

The example relies on a construction, introduced by Barge & Martin in [90], to realize the inverse limit of a given continuous map $f$ of the interval as an attractor in the plane. We will use an expanded version of this construction, which appeared later in [91], although we will not use the full generality described there (we will use the un-parameterized version). First let us recall the technique, found in [91, Section 3].

Let $M$ be a compact manifold with non-empty boundary $\partial M$. We call a subset $E \subset M$ a *boundary retract* of $M$ if there exists a continuous map $H : \partial M \times [0, 1] \to M$ that satisfies the following properties:

1. $H|_{\partial M \times [0,1)}$ is a homeomorphism onto $M \setminus E$
2. $H(x, 0) = x$ for all $x \in \partial M$
3. $H(\partial M \times \{1\}) = E$

Given a boundary retract $E \subset M$ with map $H : \partial M \times [0, 1] \to M$ there is an associated retraction $r : M \to E$ given by $r(H(x, t)) = H(x, 1)$. Finally, for a boundary retract $E \subset M$ with retraction $r : M \to E$, a map $f : E \to E$ is said to *unwrap* in $M$ if there exists a homeomorphism $F : M \to M$, called an *unwrapping*, such that $r \circ F|_E = f$ and for some $k > 0$, $F^k|_{\partial M} = id$.

The following theorem was originally proved in [90] in the case $E$ is the unit interval. The version we use here can be found in [91], although our presentation is less than general than that given in [91, Theorem 3.1] (we will not use the full version found in [91], but only require the un-parameterized construction.)

**Theorem 6.3** (Theorem 3.1 of [91]). *Let $E \subset M$ be a boundary retract with retraction $r : M \to E$, and suppose $f : E \to E$ is a continuous map which unwraps in $M$, with unwrapping $F : M \to M$. Suppose further that there exists $m > 0$ such that $f^{m+1}(E) = f^m(E)$. Then there exists a homeomorphism $G : M \to M$ such that:*

1. *$G$ has a global attractor $A$ for which $G|_A : A \to A$ is conjugate to $\tilde{f} : \varprojlim\{E, f\} \to \varprojlim\{E, f\}$*

2. *There is some $k > 0$ such that $G^k$ is the identity on $\partial M$*

Let $T : I \to I$ be the standard tent map of the unit interval $I = [0, 1]$ given by $f(x) = 2x$ for $0 \leq x \leq 1/2$, $f(x) = 2(1 - x)$ for $1/2 \leq x \leq 1$. Let $K$ denote the space obtained from the interval $I$ by identifying 0 and $2/3$ ($K$ looks like the letter $\sigma$), and let $q : I \to K$ denote the quotient map. The map $T : I \to I$ descends to a map $\overline{f} : K \to K$ since 0 and $2/3$ are fixed points of $T$.

First, to apply Theorem 6.3 we will produce an unwrapping of $\overline{f}$. Let $M$ denote the surface with boundary which is homeomorphic to an annulus (thus $M$ has two boundary components, each homeomorphic to $S^1$). We will consider $M$ as the subset of $\mathbb{R}^2$ given (in radial coordinates) as $\{(r, \theta)\ \frac{1}{2} \leq r \leq 2, 0 \leq \theta < 2\pi\}$. We first embed $K$ in to $M$ as the dashed portion in the left-hand copy of $M$ pictured in Figure 1, and letting $A$ denote the image of $[0, 2/3]$ under the quotient map $q : I \to K$, let $h : M \to M$ be the involution $h(r, \theta) = (\frac{1}{r}, \theta)$ which permutes the

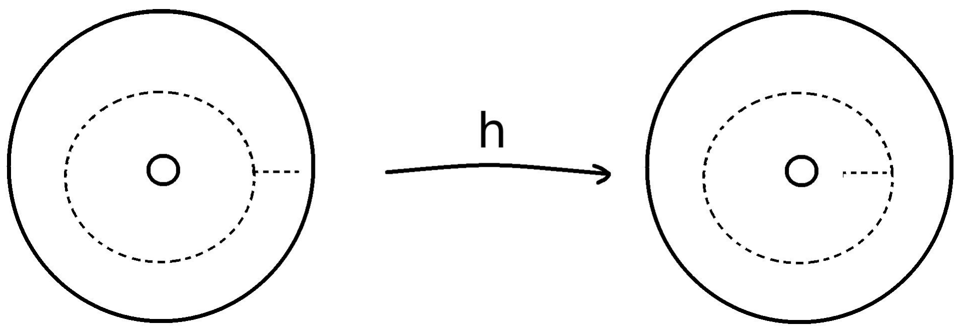


Figure 6.1: The map $h$

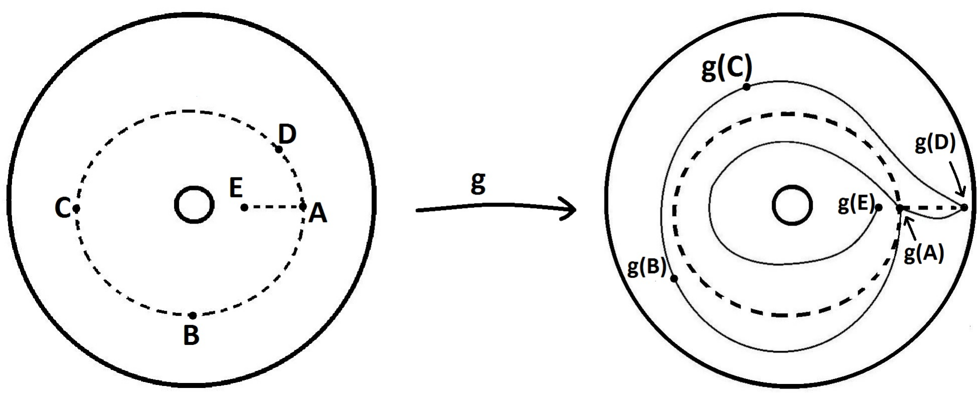


Figure 6.2: The map $g$

boundary components of $M$, and which reflects about $A$ in $M$ as pictured in Figure 1.

(Note $h$ is orientation-reversing!) We may now find a homeomorphism $g$ of $M$ such that $g$ takes $h(K)$ to the thin-solid-line graph in Figure 2.

It is clear from Figure 2 that such a $g$ can be chosen which acts by the identity on $\partial M$. Let $F = g \circ h$. By construction, $F^2 = id$ on $\partial M$, and it is not hard to see that one can choose a retraction $r : M \to K$ such that $r \circ F|_K = \overline{f}$, so $F$ is an unwrapping of $\overline{f}$, as desired.

Now let $(X, \sigma)$ be the onesided full two shift, $X = \{0,1\}^{\mathbf{N}}$ with shift map $\sigma : x_1x_2x_3 \cdots \mapsto x_2x_3 \cdots$. Let $\pi : X \to I$ be the standard Markov cover of the tent map associated with the Markov partition $\{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$, with $\pi\sigma = f\pi$. For $x = x_1x_2\cdots$, a dyadic expansion $y_1y_2\cdots$ of $\pi(x)$ can be determined recursively as follows: $y_1 = x_1$; given $y_n$, $y_{n+1} = x_{n+1}$ if $x_n = 0$, and $y_{n+1} = 1 + x_{n+1} \pmod 2$ if $x_n = 1$. All points with multiple preimages under $\pi$ are contained in the full orbit $\{f^j(1/2) \,|\, j \in \mathbb{Z}\}$ of $1/2$, the set of dyadic rationals in $I$.

Suppose $Y$ is a one-sided shift of finite type in $X$ such that the following hold:

1. $\pi^{-1}\pi Y = Y$
2. $\pi Y$ contains $\{0, 2/3\}$
3. $\pi|Y$ is injective.

Let $Z = q(\pi(Y)) \subset K$. Since $Y$ is finite type in $X$, there exists a neighborhood $U$ of $Y$ in $X$ such that $\bigcap_{n\in\mathbb{N}} f^{-n}(U) = Y$, and hence by (1), (2), and part (2) of Lemma 6.4, there exists an open $V$ in $K$ such that $Z = \bigcap_{n\in\mathbb{N}} \overline{f}^{-n}(V)$. We denote points in the inverse limit space $\varprojlim\{K, f\}$ by sequences $(x_0, x_1, x_2, \ldots) \in \prod_{i=1}^{\infty} E$ satisfying $f(x_{i+1}) = x_i$ for $i \geq 0$. Let $P : \varprojlim\{K, f\} \to K$ denote the projection $P(x_0, x_1, \ldots) = x_0$, and let $\tilde{V} = P^{-1}(V)$. Since $Z = \bigcap_{n\in\mathbb{N}} \overline{f}^{-n}(V)$, it is easy to check that $\bigcap_{n\in\mathbb{N}} \tilde{f}^{-1}(\tilde{V}) = \varprojlim\{Z, \overline{f}|_Z\}$.

Theorem 6.3 gives the existence of a homeomorphism $G : M \to M$ which contains $\varprojlim\{K, \overline{f}\}$ as a global attractor. Thus we may choose an open set $W$ in $M$ such that $W \cap \varprojlim\{K, \overline{f}\} = \tilde{V}$. We claim $W$ is an isolating neighborhood for

$\varprojlim\{Z, \overline{f}|_Z\} \subset M$. But we have $\bigcap_{n\in\mathbb{Z}} G^{-1}(W) \subset \bigcap_{n\geq 0} G^n(W) \subset W \cap \varprojlim\{K, \overline{f}\} = \tilde{V}$. Now the claim follows since $\bigcap_{n\in\mathbb{N}} \tilde{f}^{-1}(\tilde{V}) = \varprojlim\{Z, \overline{f}|_Z\}$.

Note that the system $\overline{f}|_Z$ is conjugate to the strictly sofic system obtained by identifying the two fixed points of $Y$. Thus the system $\varprojlim\{Z, \overline{f}|_Z\}$ is the inverse limit of the one-sided strictly sofic system $Z, \overline{f}|_Z$, and hence is itself a strictly sofic subshift. Finally, the map $G^2$ is orientation-preserving, contains $(\varprojlim\{Z, \overline{f}|_Z\}, G^2|_{\varprojlim\{Z,\overline{f}|_Z\}})$ as an isolated subsystem, and since $G|_{\varprojlim\{Z,\overline{f}|_Z\}}$ is strictly sofic and mixing, so will be $(\varprojlim\{Z, \overline{f}|_Z\}, G^2|_{\varprojlim\{Z,\overline{f}|_Z\}})$.

It suffices then to find such a subshift of finite type $Y$ satisfying $(1), (2)$, and (3) above. Let $Y$ be the subshift of finite type in $X$ obtained by disallowing the words 1100 and 0100. $Y$ is mixing, and $Y$ contains $0^\infty$ and $1(01)^\infty$, the unique $\pi$ preimages of 0 and 2/3. If $w$ and $x$ and distinct points in $X$ collapsed by $\pi$, then for some $n \geq 0$, $\{\sigma^n(x), \sigma^n(w)\} = \{010^\infty, 110^\infty\} = \pi^{-1}(1/2)$. Then conditions (1) and (3) follow since the words 1100 and 0100 are not allowed in $Y$. Thus $Y$ satisfies (1)-(3) as required.

We note that the strictly sofic system isolated in this example is obtained from a shift of finite type by identifying precisely two points. We do not know how to get an example collapsing more than finitely many points.

## 6.8 Isolation and flow equivalence in dimension two

Recall two compact zero-dimensional systems $(X, f), (Y, g)$ are said to be *flow equivalent* if there exists a homeomorphism $h : \Sigma_f X \to \Sigma_g Y$ between their suspensions such that $h$ preserves the directions of the flow. We will show that for a compact zero-dimensional system $(X, f)$, whether $(X, f)$ is orientably isolatable in dimension two depends only on the flow equivalence class of $(X, f)$.

The following result relies on some consequences of the Jordan-Schoenflies Theorem, due to Moise in [85, Theorem 13.1]. We call $E$ a *2-cell* if $E$ is homeomorphic to the closed unit 2-disk in $\mathbb{R}^2$, and by an *open 2-cell* we mean a space homeomorphic to the open unit 2-disk.

**Theorem 6.4.** *[85, Theorem 13.1]*

1. *Let $E$ be 2-cell, $n \geq 1$, and let $\{A_i\}_{i=1}^{2n}$ be a collection of pairwise disjoint 2-cells each contained in* $\mathrm{Int}(E)$. *There exists a homeomorphism $f : E \to E$ such that $f|_{\partial E} = id$, and for all $1 \leq i \leq n$, $f(A_i) = A_{i+n}$ and $f(A_{n+i}) = A_i$.*

2. *Let $M$ be an orientable two-manifold with orientation-preserving homeomorphism $f : M \to M$, and suppose $\{A_i\}_{i=1}^{n}$ is a collection of pairwise disjoint 2-cells in $M$ such that $f(A_i) = A_i$ for all $1 \leq i \leq n$. Then there exists a collection of 2-cells $\{B_i\}_{i=1}^{n}$ such that $B_i \subset A_i$ for all $1 \leq i \leq n$, and a homeomorphism $g : M \to M$ such that, for all $1 \leq i \leq n$, $g|_{M \setminus \cup_{i=1}^{n} \mathrm{Int} A_i} = f|_{M \setminus \cup_{i=1}^{n} \mathrm{Int} A_i}$ and $g|_{B_i} = id$. Moreover, given a collection of compact sets $\{Y_i\}_{i=1}^{n}$ such that $Y_i \subset \mathrm{Int} A_i$ for all $1 \leq i \leq n$, the collection $B_i$ may be chosen such that*

$Y_i \subset \mathrm{Int}B_i$ *for all* $1 \leq i \leq n$.

*Proof.* Part (1) follows from [85, Theorem 13.1]. For part (2), suppose we have such a collection of compact sets $Y_i \subset \mathrm{Int}A_i$. Since $f$ is orientation-preserving the homeomorphism $f|_{\partial A_1} : \partial A_1 \to \partial A_1$ is isotopic to the identity. Let $h : A_1 \to \mathbb{D}$ be a homeomorphism taking $A_1$ to the unit disk $\mathbb{D} = \{(r, \theta) \,|\, 0 \leq r \leq 1, 0 \leq \theta < 2\pi\}$, and choose $\epsilon > 0$ such that $h(Y_1) \subset \{(r, \theta) \in \mathbb{D} \,|\, r < 1 - \epsilon\}$. Then $h|_{\partial A_1} \circ f|_{\partial A_1} \circ h^{-1}|_{\partial \mathbb{D}}$ is isotopic to the identity as well, and we let $H : S^1 \times I \to S^1$ denote such an isotopy with $H|_{S^1 \times \{1\}} = id$. Define $j_1 : \mathbb{D} \to \mathbb{D}$ by $j_1(r, \theta) = H(\theta, (1 - r)2/\epsilon)$ for $\{(r, \theta) \,|\, 0 \leq \theta < 2\pi, 1 - \epsilon/2 \leq r \leq 1\}$, and $j_1 = id$ on the remainder of $\mathbb{D}$. Finally, let $g_1 = h^{-1} j_1 h$. Note that $g_1$ acts as the identity on the set $h^{-1}(\{(r, \theta) \,|\, 0 \leq \theta < 2\pi, 1 - \epsilon/2 \leq r \leq 1\})$, which contains $Y_1$ in its interior.

Since the preceding construction only took place on $A_1$, we may repeat on each $A_i$, and extend as the identity outside of $\cup_{i=1}^{n} A_i$ to construct $g$. □

**Lemma 6.5.** Let $f : M \to M$ be an orientation-preserving homeomorphism of an orientable surface $M$ for which $X \subset M$ is a compact zero-dimensional isolated invariant set under $f$. Then there exists a homeomorphism $g : M \to M$ for which $X$ is isolated invariant under $g$, with an isolating neighborhood $U$ for $X$ under $g$ such that there is a disc $D$ in $M \setminus U$ with $g|D = id$.

*Proof.* Let $U$ be an isolating neighborhood for $X$. Since $X$ is compact zero-dimensional, we may assume, by choosing a smaller isolating neighborhood if necessary, that each component of $U$ is an open 2-cell, and for which there exists $z \in M \setminus U$ such that $f(z) \in M \setminus U$. (If there were no such $z$, this would imply that $(M \setminus U) \cap f^{-1}(M \setminus U) =$

$\emptyset$, i.e. $M = U \cup f^{-1}(U),$). We may furthermore assume $z$ is such that $z \neq f(z)$.

We claim there is a 2-cell $V$ disjoint from $U$, which contains $z$ and $f(z)$ in its interior. This is not hard to construct, since $M \setminus U$ is path-connected, using a tubular neighborhood for example. We may now choose a 2-cell $D$ contained in $\mathrm{Int}(V)$ containing $z$ such that $f(D)$ is disjoint from $D$ and $f(z) \in f(D) \subset \mathrm{Int}(V)$. Part (1) of Theorem 6.4 now implies there exists a homeomorphism $h : V \to V$ which is the identity on the boundary of $V$, for which $h(D) = f(D)$, and $h(f(D)) = D$. Extend $h$ to a homeomorphism, which we also denote by $h$, to all of $M$ by acting via the identity on $M \setminus V$. By part (2) of Theorem 6.4, we may find $h_1$ and a disk $E \subset D$ for which $h_1|_E = id$ and $h|_{M \setminus V} = id$. Now let $g = h_1 \circ f$. Clearly $D$ is invariant under $g$. We claim that $X$ is still isolated invariant under $g$, with the isolating neighborhood $U$. Indeed, suppose $x \in U \setminus X$. Then there exists a least $|n|$ such that $f^n(x) \in M \setminus U$. If $n > 0$, then $g^n(x) = h_1 \circ f \circ f^{n-1}(x) = h_1 \circ f^n(x) \in M \setminus U$, since $h_1$ acts via the identity on $M \setminus V$, which contains $U$. If $n < 0$, then $g^n(x) = f^{-1} \circ h_1^{-1} \circ g^{n+1}(x) = f^{-1} \circ f^{n+1}(x) = f^n(x) \in M \setminus U$, again since $h_1$ acts via the identity on $U \subset M \setminus V$. It follows that $X$ is still isolated invariant under $g$. $\square$

We are now in a position to prove the following theorem, showing that, for compact zero-dimensional systems, being orientably isolatable in dimension two depends only on the flow equivalence class.

**Theorem 6.5.** *Suppose $(X, f)$ and $(Y, g)$ are two flow equivalent compact zero-dimensional systems, and suppose $(X, f)$ is orientably isolatable in dimension two.*

*Then $(Y, g)$ is also orientably isolatable in dimension two.*

To prove Theorem 6.5 we will need the following characterization of flow equivalence of compact zero-dimensional systems, due to Parry and Sullivan [92].

**Definition 6.6.** Given a homeomorphism of a compact zero-dimensional space $f : X \to X$ and a decomposition $X = A \cup B$ in to disjoint clopen subsets, an *expansion* $\tilde{f}$ of $f$ (along $A$) is defined as follows: let $j : A \to \tilde{A}$ denote a homeomorphism between $A$ and a copy $\tilde{A}$ of $A$, let $\tilde{X} = A \cup \tilde{A} \cup B$, and define $\tilde{f} : \tilde{X} \to \tilde{X}$ as follows: $\tilde{f} = j$ on $A$, $\tilde{f} = f \circ j^{-1}$ on $\tilde{A}$, and $\tilde{f} = f$ on $B$.

**Theorem 6.6.** *[92] The flow equivalence relation on homeomorphisms of compact zero-dimensional spaces is generated by topological conjugacy and expansion.*

*Proof of Theorem 6.5.* It is clear that if $(X, f)$ and $(Y, g)$ are conjugate and $(X, f)$ is orientably isolatable in dimension two, then so is $(Y, g)$. Thus by Theorem 6.6 it is enough to prove the following statement:

(2.2.1) If $(X, f)$ is a compact zero-dimensional system and $(\tilde{X}, \tilde{f})$ is an expansion of $(X, f)$, then $(\tilde{X}, \tilde{f})$ is orientably isolatable in dimension two if and only if $(X, f)$ is.

We first prove the forward direction. To this end, suppose $(X, f)$ is isolated in the orientably two-manifold $M$ under the orientation-preserving homeomorphism $h : M \to M$. Let $A \subset X$ be a proper clopen subset, $B = X \setminus A$, and suppose $Z$ is an isolating neighborhood for $X$. By applying Lemma 6.5 we may assume $X$ has an isolating neighborhood $Z_1$ such that $h$ has a 2-cell $D$ disjoint from $\overline{Z_1}$ for which $h|_D = id$. Since $X$ is compact and zero-dimensional, we may then further find a neighborhood $U$ of $X$ with $U \subset Z_1$ such that, if $\{U_i\}$ denotes the connected components of $U$, we have

$$\Big( \bigcup_{U_i \cap A \neq \emptyset} U_i \Big) \cap \Big( \bigcup_{U_i \cap B \neq \emptyset} U_i \Big) = \emptyset$$

and such that each $U_i$ is a 2-cell. Note that since $U \subset Z_1$ and $Z_1$ is an isolating neighborhood, $U$ is an isolating neighborhood of $X$.

We claim there exists a 2-cell $V$ satisfying each of the following:

$$V \cap \bigcup_{U_i \cap B \neq \emptyset} U_i = \emptyset, \qquad \bigcup_{U_i \cap A \neq \emptyset} U_i \subset V, \qquad D \subset V$$

To see this, let us first enumerate the collection $\{U_i \,|\, U_i \cap A \neq \emptyset\}$ by $\{U_i\}_{i=1}^{k}$. Since each component of $U$ is a 2-cell, we may find a simple curve $\alpha$ such that $U_1 \cup U_2 \cup \alpha$ is path connected and disjoint from $\bigcup_{U_i \cap B \neq \emptyset} U_i$, and thickening this curve to a 2-cell, thereby find a 2-cell containing $U_1 \cup U_2$. Continuing this procedure inductively, we may find a 2-cell $V$ containing $\bigcup_{U_i \cap A \neq \emptyset} U_i \cup D$ with $V \cap \bigcup_{U_i \cap B \neq \emptyset} U_i = \emptyset$. Now once again using that $X$ is compact and zero-dimensional, we select disjoint 2-cells $\{E_i\}_{i=1}^{m}$ satisfying, for all $i$:

$$E_i \subset \mathrm{Int}(V), \qquad A \subset \mathrm{Int}(\bigcup_{i=1}^{m} E_i), \qquad E_i \cap D = \emptyset$$

Choose $m$ disjoint 2-cells $\{F_i\}_{i=1}^{m}$ each contained in $\mathrm{Int}(D)$. It follows from part (1) of Theorem 6.4 that there exists a homeomorphism $g : V \to V$ which is the identity on the boundary of $V$, for which $g$ maps $E_i$ homeomorphically onto $F_i$ and $F_i$ homeomorphically onto $E_i$, for all $1 \leq i \leq m$. Using part (2) of Theorem 6.4, we may without loss of generality assume the collection $\{E_i\}_{i=1}^{n}$ and $g$ are such that

$g^2|_{E_i} = id$ for all $i$, with $A \subset \mathrm{Int}(\bigcup_{i=1}^m E_i)$. Extend $g$ to all of $M$ by letting it act via the identity outside of $V$. Finally, let $H = h \circ g$. Let $\tilde{X} = X \cup g(A)$. Note that $\tilde{X}$ is invariant under $H$, and $(\tilde{X}, H)$ is conjugate to the extension of $(X, f)$ along $A \subset X$. We will show that $\tilde{X}$ is also isolated invariant under $H$.

Let $U_B$ denote the union of the components of $U$ which intersect $B$, let $U_A = \cup_{i=1}^n E_i$, and let $U_{g(A)} = g(U_A)$. We claim that $\tilde{U} = U_A \cup U_B \cup U_{g(A)}$ is an isolating neighborhood for $\tilde{X}$. To see this, suppose instead there exists $y \in \tilde{U}$ such that $H^n(y) \in \tilde{U} \setminus \tilde{X}$ for all $n \in \mathbb{Z}$. Let $I_y = \{i \in \mathbb{Z} \,|\, H^i(y) \in U_A \cup U_B\}$. Let $x = y$ if $y \in U_A \cup U_B$, and $x = g(y)$ if $y \in U_{g(A)}$. It follows from the construction of $H$ that $\mathcal{O}_h(x)$, the orbit of $x$ under $h$, agrees with the set $\{H^i(y) \,|\, i \in I_y\}$. Thus $h^n(x) \in U_A \cup U_B \setminus X$ for all $n \in \mathbb{Z}$, contradicting the isolation of $X$ under $h$ by $U_A \cup U_B$. This completes the proof of one direction of (2.2.1).

To complete the proof of Theorem 6.5, it suffices to prove the following: if $f : X \to X$ is a homeomorphism of a compact zero-dimensional space for which there exists an expansion $\tilde{f} : Y \to Y$ of $(X, f)$ such that $(Y, \tilde{f})$ is orientably isolatable in dimension two, then $(X, f)$ is orientably isolatable in dimension two. Suppose we are in this scenario, with a homeomorphism $g : M \to M$ having an isolated invariant set $Y$ such that $g|_Y$ is an extension of $(X, f)$ using the clopen subset $A \subset X$. By the above, the extension of $g|_Y$ along $X \setminus A \subset X \subset Y$ is orientably isolatable in dimension two. This extension is conjugate to the system $(\Gamma, h)$, where $\Gamma$ is the disjoint union of two copies of $X$, say $X_1 = X, X_2 = X$, $\Gamma = X_1 \cup X_2$, and $h : X_1 \to X_2$ is the identity while $h : X_2 \to X_1$ is the map $f$.

With this in mind, it is sufficient to prove the following lemma.

**Lemma 6.7.** Let $f$ be a homeomorphism of a surface $M$ for which $X \subset M$ is a compact zero-dimensional isolated invariant set under $f$. Suppose further that $X$ has a partition into clopen subsets $X = A \cup B$, such that $f(A) = B$, $f(B) = A$, and let $g = f^2|_A : A \to A$. Then $(A, g)$ is isolated under $f^2 : M \to M$.

To prove the lemma, we will use the notion of filtration pairs found in [93]. We recall the definition, and the relevant result from [93].

**Definition 6.8.** Let $f : M \to M$ be a homeomorphism of a manifold, and $X$ an isolated invariant set under $f$. A compact pair $L \subset N$ is called a *filtration pair* for $X$ if $N$ and $L$ are closures of their interiors, and satisfy:

1. $\overline{N \setminus L}$ is an isolating neighborhood for $X$.

2. $L$ is a neighborhood in $N$ of the set $N^- = \{x \in N \,|\, f(x) \notin \mathrm{Int}(N)\}$.

3. $f(L) \cap \overline{N \setminus L} = \emptyset$.

**Theorem 6.7.** *[93, 3.7] If $Y$ is an isolated invariant set for a homeomorphism $f : M \to M$ of a manifold $M$, then every neighborhood for $Y$ contains a filtration pair $(N, L)$.*

In fact, it is shown in [93, 3.7] that $N$ may be chosen to be a compact manifold with boundary, a fact which we will not need.

*Proof of Lemma 6.7.* Since $X$ is isolated, by Theorem 6.7, there exists a filtration pair $(N, L)$ for $X$. We claim that

$$f(\overline{N \setminus L}) \cap f^{-1}(\overline{N \setminus L}) \cap (M \setminus (\overline{N \setminus L})) = \emptyset$$

To see this, first note that by condition (3) in the definition of filtration pair, we have $f(L \cap f^{-1}(\overline{N \setminus L})) = f(L) \cap (\overline{N \setminus L}) = \emptyset$, so $L \cap f^{-1}(\overline{N \setminus L}) = \emptyset$. Thus

$$f^{-1}(\overline{N \setminus L}) \cap (M \setminus (\overline{N \setminus L})) \subset f^{-1}(\overline{N \setminus L}) \cap M \setminus N \tag{6.9}$$

since $M \setminus \overline{N \setminus L} \subset (M \setminus N) \cup L$. Since $L$ is a neighborhood of $\{x \in N \,|\, f(x) \notin \mathrm{Int}(N)\}$ in $N$, we have $f(\overline{N \setminus L}) \cap (M \setminus N) = \emptyset$. Thus we get, using (6.9),

$$f(\overline{N \setminus L}) \cap f^{-1}(\overline{N \setminus L}) \cap (M \setminus (\overline{N \setminus L})) \subset f(\overline{N \setminus L}) \cap f^{-1}(\overline{N \setminus L}) \cap M \setminus N = \emptyset$$

proving the claim.

Since $\overline{N \setminus L}$ is a neighborhood of the clopen sets $A, B$, we may choose a compact neighborhood $U$ of $B$ such that $U \subset N \setminus L$, and $A \subset \mathrm{Int}(\overline{N \setminus L} \setminus U)$. Let $V = \overline{N \setminus L} \setminus U$. Note that $V \subset \overline{N \setminus L}$, and $V$ is a neighborhood of $A$.

We claim that $V$ is an isolating neighborhood for $A$ under $f^2$. Indeed, suppose $x \in V \setminus A$. Then $x \in \overline{N \setminus L} \setminus B$, so there exists $j \in \mathbb{Z}$ such that $f^j(x) \in M \setminus (\overline{N \setminus L})$. But now since $f(\overline{N \setminus L}) \cap f^{-1}(\overline{N \setminus L}) \cap (M \setminus (\overline{N \setminus L})) = \emptyset$, we must have either $f^{j-1}(x) \in M \setminus (\overline{N \setminus L})$ or $f^{j+1}(x) \in M \setminus (\overline{N \setminus L})$. Since $V \subset M \setminus (\overline{N \setminus L})$, this proves the claim. $\square$

$\square$

## 6.9 A cohomological obstruction to isolation in dimension two

This section presents obstructions for certain compact zero-dimensional systems to be isolated by a surface homeomorphism. Recall a zero-dimensional system $(X, f)$ is called *indecomposable* if the only $f$-invariant clopen subsets are either $X$ or $\emptyset$. For a zero-dimensional system $(X, f)$, the suspension system $\Sigma_f X$ is a compact metric space locally homeomorphic to the product of $X$ with an arc. A zero-dimensional system $(X, h)$ is indecomposable if and only if its suspension $\Sigma_h X$ is connected. The obstructions we consider in this section for an indecomposable zero-dimensional system to be isolatable are concerned with the structure of the abelian group $\check{H}^1(\Sigma_f X, \mathbb{Z})$, the first Čech cohomology of the suspension.

A space $X \subset \mathbb{R}^n$ is called a Euclidean neighborhood retract (ENR) if $X$ is a retract of some neighborhood of $X$ in $\mathbb{R}^n$.

**Isolation for flows in dimension 3:** Suppose now that $M$ is a compact 3-manifold, with a continuous flow $\phi : M \times \mathbb{R} \to M$. For $S \subset \mathbb{R}$ and $A \subset M$, we sometimes denote $\phi(A \times S)$ by $A \cdot S$. Similar to the discrete case, a compact invariant subset $I \subset M$ is *isolated* if it is the maximal invariant set in some neighborhood of itself, and such a neighborhood is called an *isolating neighborhood*. For $\Sigma \subset M$ and $\delta > 0$, if $\phi_\delta : \Sigma \times (-\delta, \delta) \to M$ is a homeomorphism onto its image with open range, then $\Sigma$ is called a *local section*. In this case, the set $Image(\phi)$ is called a *collar* of $\Sigma$.

Suppose $B \subset M$ is a closed subset, $\Sigma^+$, $\Sigma^-$ are local sections with disjoint closures, and choose $\delta > 0$ such that $\phi_\delta(\Sigma^+ \times (-\delta, \delta))$ and $\phi_\delta(\Sigma^- \times (-\delta, \delta))$ are collars of $\Sigma^+$, $\Sigma^-$, respectively. $B$ is called an *isolating block* if

1. $(Cl(\Sigma^\pm) \setminus \Sigma^\pm) \cap B = \emptyset$

2. $\Sigma^+ \cdot (-\delta, \delta) \cap B = (\Sigma^+ \cap B) \cdot [0, \delta)$ and $\Sigma^- \cdot (-\delta, \delta) \cap B = (\Sigma^- \cap B) \cdot (-\delta, 0]$

3. for $p \in (\partial B \setminus (\Sigma^+ \cup \Sigma^-))$ there exist $\epsilon_1 < 0 < \epsilon_2$ such that

   $p \cdot \epsilon_1 \in \Sigma^+, p \cdot \epsilon_2 \in \Sigma^-$, and $p \cdot [\epsilon_1, \epsilon_2] \subset \partial B$

Note that isolating blocks are also isolating neighborhoods. For such a block $B$, let $b^+ = \Sigma^+ \cap \partial B$ and $b^- = \Sigma^- \cap \partial B$. It follows from (1) that $b^+, b^-$ are always closed. Given $B$, we also define $A^+ = \{p \in B | \phi(\{p\} \times [0, \infty)) \subset B\}$, $A^- = \{p \in B | \phi(\{p\} \times (-\infty, 0] \subset B\}$, and let $a^\pm = A^\pm \cap b^\pm$. For any such block $B$, $a^\pm$ are closed and satisfy $a^\pm \subset Int(b^\pm)(rel \Sigma^\pm)$ [94, Prop. 3.7].

The following theorem guaranteeing the existence of isolating blocks was proved in [94, Theorem 3.4]; the portion regarding the block being an ENR with $b^-$ a manifold is from [95, Theorem 1.2].

**Theorem 6.8.** *(Churchill, Ruchala) If $I$ is an isolated invariant set in a flow on a compact 3-manifold, any isolating neighborhood contains an isolating block $B$ which isolates $I$. Furthermore, $B$ may be chosen to be an ENR, such that $b^-$ is a topological 2-manifold.*

A key part in the proof of Theorem 6.8 showing that $B$ is an ENR is that $\Sigma^-$ is a topological 2-manifold, which follows from the fact that $\Sigma^-$ is a local section, along with a theorem of Borsuk [96, Theorem 13] showing that if $X$ is two-dimensional and a topological divisor of an open subset of $\mathbb{R}^n$ (i.e. there exists $Y$ such that

$X \times Y$ is homeomorphic to an open subspace of $\mathbb{R}^n$) for some $n$, then $X$ is locally homeomorphic to $\mathbb{R}^2$.

Suppose now that $I$ is a compact connected invariant set, isolated by a block $B$. By Theorem 6.8 we may assume that $B$ is a compact ENR and $b^-$ is a compact topological 2-manifold with boundary; since $I$ is connected, we may assume $B$ is connected. The following theorem of Thomas was proved in the setting of $C^1$ flows for the case $I$ is minimal. Although [97] concerns smooth flows, as he mentions, the continuous case can be obtained using the concept of a topological isolating block. The proof in [97] carries over to our setting, with the important ingredient that for an isolating block $B$ the map $\sigma^- : B \to [0, \infty]$ defined by $\sigma^-(p) = \sup\{t \geq 0 | p \cdot [0,t] \cap \Sigma^- = \emptyset\}$ for $p \in B \setminus \Sigma^-$ and $\sigma^-(p) = 0$ if $p \in \Sigma^-$, is continuous. The continuity of $\sigma^-$ and $\sigma^+$ is proved in [94, Proposition 3.1].

**Theorem 6.9.** *[97, Theorem 2] The sequence $\check{H}^q(B) \xrightarrow{i^*} \check{H}^q(I) \xrightarrow{j} \check{H}^{q+1}(b^-, a^-)$ is exact for all $q \geq 1$.*

Here the map $i^*$ is the map induced by the inclusion $i : I \hookrightarrow B$. Since $B$ is a compact ENR, $\check{H}^p(B)$ is finitely generated for all $p$ ( [98, Corollary A.8]), so in particular, the image of $i^*$ is finitely generated.

**Lemma 6.10.** [97, pg. 241] $\check{H}^2(b^-, a^-)$ is isomorphic to $F \oplus T$ where $F$ is free and $T$ is finite.

We note that in the case $M$ is orientable, Lemma 6.10 can be deduced immediately using duality. Indeed, if $M$ is orientable then $\Sigma^-$ must be orientable as well, since $\Sigma^-$ is a local section. Since $b^-$ and $a^-$ are compact, $a^- \subset Int(b^-)(rel\Sigma-)$, and $\Sigma^-$ is an orientable two-manifold, duality gives $\check{H}^2(b^-, a^-) \cong H_0(\Sigma^- - a^-, \Sigma^- - b^-)$ (see [99, Proposition 7.2], which is free.

For an abelian group $H$, an element $a \in H$ is said to have *infinite height* if there exists integers $n_i \to \infty$ such that $n_i x = a$ has a solution in $H$ for all $i$. We say an abelian group $H$ has *finite height* if $H$ has no non-zero elements of infinite height.

**Theorem 6.10** (Thomas)**.** *Suppose $I$ is a compact connected invariant set isolated by a flow $\phi : M \times \mathbb{R} \to M$ where $M$ is a compact 3-manifold. Then $\check{H}^1(I)$ has finite height.*

*Proof.* We continue using the notation introduced previously. We first note at least one of $b^-$ or $b^+$ is empty; we will assume $b^-$ is non-empty, as the argument can easily be adapted to the other case. Thus we will assume $b^-$ is non-empty. The remainder of the proof is essentially the argument given in [97, pg. 241]. By Theorem 6.9, we have the exact sequence $\check{H}^1(B) \xrightarrow{i^*} \check{H}^1(I) \xrightarrow{j} \check{H}^2(b^-, a^-)$. If $x \in \check{H}^1(I)$ has infinite height, then its image in $H^2(b^-, a^-)$ has infinite height, and hence lies in the finite subgroup $\gamma^{-1}(T) \subset \check{H}^2(b^-, a^-)$, where $\gamma : \check{H}^2(b^-, a^-) \to F \oplus T$ is the isomorphism coming from Lemma 6.10. Thus $kx \in ker(j)$ for some $k$. Note that the element $kx$ also has infinite height. Since $\check{H}^1(I)$ is torsion-free, if $x \neq 0$, then $ker(j)$ is not finitely generated, since the subgroup generated by $kx$ would not finitely generated. However $B$ is a compact ENR, so $\check{H}^p(B)$ is finitely generated for all $p$ ( [98, Corollary

A.8]), and in particular, the image of $i^*$ is finitely generated. Exactness then gives a contradiction, and we must have $x = 0$. $\square$

Let us summarize how Theorem 6.10 gives obstructions for a compact zero-dimensional indecomposable system $(X, f)$ to be isolatable in dimension two. If the system $(X, f)$ is isolated by a surface homeomorphism $h : M \to M$, its suspension $\Sigma_h M$ is a 3-manifold containing $\Sigma_f X$ as a compact connected isolated invariant set. Theorem 6.10 then implies $\check{H}^1(\Sigma_f X)$ has finite height.

**Coinvariants:** We will now show how the cohomological obstruction presented in the previous section can be restated in a way which is more intrinsic to the zero-dimensional system $(X, f)$.

For a homeomorphism $f$ of a compact zero-dimensional space $X$ let $C(X, \mathbb{Z})$ be the abelian group of continuous functions $g : X \to \mathbb{Z}$ and define $\partial : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ by $\partial(g) = g - g \circ f$. We let $G_{X,f} := C(X, \mathbb{Z})/Image(\partial)$ denote the quotient group, often called the group of coinvariants. It is a classic fact that for a compact zero-dimensional system $(X, f)$, the abelian group $G_{X,f}$ is isomorphic to the first Cech cohomology of the suspension $\check{H}^1(\Sigma_f X, \mathbb{Z})$ (see [100, Chapter IV Section 3]).

**Theorem 6.11.** *If $(X, f)$ is a zero-dimensional compact isolated indecomposable invariant set of a surface homeomorphism $h : M \to M$, then $G_{X,h|_X}$ has no elements of infinite height.*

*Proof.* If $(X, f)$ is isolated, it's suspension $\Sigma_f X$ is connected and is isolated by the flow on $\Sigma_h M$. The result then follows from the fact that $\check{H}^1(\Sigma_f X) \cong G_{X,f}$, along with Theorem 6.10. ☐

*Proposition* 6.11. Suppose $(X, h)$ is an indecomposable extension of a Cantor system $(Y, g)$ for which $\mathbf{1} \in G_{Y,g}$ has infinite height, where $\mathbf{1}$ denotes the function $\mathbf{1}(x) = 1$ for all $x$. Then $(X, h)$ can not be isolated in dimension two.

*Proof.* If $\pi : X \to Y$ denotes the given factor map, there is an induced homomorphism $\pi^* : G_{Y,g} \to G_{X,h}$ taking $0 \neq [\mathbf{1}]$ in $G_{Y,G}$ to $[\mathbf{1}] \neq 0$ in $G_{X,h}$. Since $\mathbf{1}$ in $G_{Y,g}$ is of infinite height, so is $\pi^*(\mathbf{1})$, and the result follows from Theorem 6.11. ☐

*Example* 6.12. Let $(X_n, f)$ denote the standard $n$-odometer system, $n \geq 2$. For example we may let $X_n$ be the group $X_p = \varprojlim\{\mathbb{Z}/n^k, \pi_k\}$ where $\pi_k : \mathbb{Z}/n^{k+1} \to \mathbb{Z}/n^k$ is the natural projection map, and $f : X_n \to X_n$ by given by $f(x) = x + 1$, where 1 denotes the element $(1, 1, \ldots) \in X_n$. $(X_n, f)$ is a minimal Cantor system (see [101]), and it is not hard to show that $G_{X_n,f} \cong \mathbb{Z}[1/n]$, a group which consists of elements of infinite height. Since $(X_n, f)$ is minimal (and hence indecomposable), it follows that the standard odometers can not arise as an isolated invariant set for a surface homeomorphism. Corollary 6.11 also rules out any extension $(Y, g)$ of an odometer, such as those in the following Corollary.

**Corollary 6.12.** *Any Toeplitz system is a minimal extension of an odometer (see [101, pg. 14]), so can not be isolated in dimension two.*

In general, minimal Cantor systems $(X, h)$ for which $G_{X,h}$ contains an element of infinite height are abundant. Apart from odometers and Toeplitz systems, many examples can be constructed using substitution systems, which we briefly recall. For a finite alphabet $\mathcal{A}$ with $|\mathcal{A}| \geq 2$, a substitution is a map $\tau : \mathcal{A} \to \cup_{n=1}^{\infty} A^n = A^*$. By a substitution system we mean the two-sided shift map $\sigma$ on the set $X_\tau \subset \mathcal{A}^{\mathbb{Z}}$ of bi-infinite sequences whose finite sub-words all belong to the language of a substitution $\tau : \mathcal{A} \to \mathcal{A}^*$ on the finite alphabet $\mathcal{A}$. If $\tau$ is primitive (there exists $n$ such that for all $a \in \mathcal{A}$, $\tau^n(a)$ contains every letter from $\mathcal{A}$), then $(X_\tau, \sigma)$ is minimal (see [102, 1.2]). The abelianization matrix $M_\tau$ associated to a substitution $\tau$ is defined by setting $M_\tau(i,j)$ equal to the number of occurrence of the letter $i$ in $\tau(j)$. If the product of the non-zero eigenvalues of $M_{tau}$ is not $\pm 1$, then $G_{X_\tau,\sigma}$ contains non-zero elements of infinite height. More general constructions for Cantor systems whose group of coinvariants contains non-zero elements of infinite height can be given using Bratteli diagrams, for which we refer the reader to [103].

**Remark 1:** It is worth noting that for any countable index set $I$, $\bigoplus_{i \in I} \mathbb{Z}$ can be realized as the group of coinvariants of a Cantor dynamical system which appears as an isolated invariant set in dimension 2. Indeed, for $1 \leq n \leq \aleph_0$, [104] give a construction of a Denjoy homeomorphism of $S^1$ with a unique minimal Cantor set $\Sigma$ such that $K_0(C(\Sigma) \rtimes_{\varphi_n} \mathbb{Z}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}$. For such systems, $K_0(C(\Sigma) \rtimes_{\varphi_n} \mathbb{Z})$ is isomorphic to the group of coinvariants of $(\Sigma, \varphi_n)$ [105]. $\varphi_n$ can then be isolated in dimension 2 using Theorem 2.

## 6.10 Odometers in dimension two are limits of periodic points

Let $(X_n, f)$ denote the standard $n$-odometer system. The system $(X_n, f)$ is a minimal equicontinuous system (in fact, these characterize odometers - see [106].) Since the group of coinvariants for an odometer has elements of infinite height, Theorem 6.11 implies they can not be isolated in dimension 2. The goal of this section is to prove a companion theorem: if an odometer is an invariant set in dimension two, it must be the limit of periodic points.

This generalizes Theorem 2 in [107], since we do not require the invariant set to be stable (in the sense of [107]), nor are we necessarily in the plane. That an odometer can even appear as an invariant set of a homeomorphism in dimension 2 is clear from Theorem 3, or one can find a particular construction in [107]. We first record the following elementary property of odometers, which will be useful in the proof.

**Lemma 6.13.** Let $(X, f)$ be an odometer system. There exists a sequence of compact subsets $U_i$ and natural numbers $n_i$ such that $f^{n_i}(U_i) = U_i$ for all $i$, and $diam(U_i) \to 0$ as $i \to \infty$.

For a space $X$ with homeomorphism $f : X \to X$, let $P_X$ denote the set of periodic points of $X$ under $f$.

**Theorem 6.13.** *Suppose $M$ is a connected two-manifold without boundary such that $X$ is an invariant set for a homeomorphism $h : M \to M$, and $h|_X$ is conjugate to an odometer. Then $X \subset \overline{P_M}$.*

*Proof.* We first suppose $P \neq \emptyset$, and treat the case $P = \emptyset$ later. Suppose then instead that $\overline{P} \cap \Gamma = \emptyset$, let $X = M \setminus \overline{P}$. Let $\{U_i\}, n_i$ be the sequence of compact subsets as given by the lemma, and let $A$ be a component of $M \setminus \overline{P}$ containing infinitely many of the $U_i$ - by relabeling, call these $U_i$. Then $A$ is a non-compact connected two-manifold without boundary, and hence has a universal cover $\tilde{A}$ homeomorphic to $\mathbb{R}^2$. Lift $\Gamma_A = A \cap \Gamma$ to disjoint copies $\tilde{\Gamma_A}^{(k)}$ in $\tilde{A}$. Choose $\epsilon > 0$ such that different copies of $\tilde{\Gamma_A}^{(k)}$ are bounded away by $\epsilon$. Equicontinuity then gives a $\delta$ such that two points $\delta$ close remain $\epsilon$ close under any power of $h$. Now choose $n_i$ such that $h^{n_i}(U_i) = U_i$ with $diam(U_i) < \delta$, and let $\tilde{U}_i^{(k)}$ denote the different copies of the lifted $U_i$. Then $h^{n_i}$ maps $A$ to itself, and any lift $\tilde{h}^{n_i}$ of $h^{n_i}$ must satisfy $\tilde{h}^{n_i}(U_i^{(k)}) = \tilde{U}_i^{(j)}$, hence we can choose a lift $\tilde{h}^{n_i}$ fixing a $\tilde{U}_i^{(k)}$. However, now $\tilde{h}^{n_i}$ (or replacing with $\tilde{h}^{2n_i}$ if $h$ is not orientation preserving) is an orientation preserving homeomorphism of $\mathbb{R}^2$ with a non empty invariant compact set, and hence has a nonwandering point. But Brouwer's Nonwandering theorem then implies $\tilde{h}^{n_i}$ has a fixed point, a contradiction.

Now suppose $P = \emptyset$. Then $M$ has a universal cover which is homeomorphic to either $\mathbb{R}^2$ or $S^2$. If the universal cover is $\mathbb{R}^2$, the above argument applies and shows $P \neq \emptyset$, a contradiction. If the universal cover is $S^2$, then one can again lift $h^{n_i}$ to an orientation preserving homeomorphism $\tilde{h}^{n_i}$ of the universal cover, and topological reasons implies such a homeomorphism must have a fixed point however, again contradicting $P = \emptyset$.

Finally, suppose $x \in X$. By the above there exists $y \in X$ and a sequence $p_i \in P$ such that $p_i \to y$. Choosing a sequence $n_i$ such that $h^{n_i}(y) \to x$, we have $f^{n_i}(p_i) \to x$, while $f^{n_i}(p_i) \in P$ for all $i$. Thus $x$ is a limit of periodic points, and the proof is complete.

$\square$

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